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New Challenges in Unification

- Higher Dimensional Unified Theories and Coset Space Dimensional Reduction
- Fuzzy Extra Dimensions and Renormalisable Chiral Unified Theories
- Remarks on the predictions of Finite Unified Theories vs LHC results

Coset Space Dimensional Reduction
(CSDR)

Original motivation

Use higher dimensions

- to unify the gauge and Higgs sectors
- to unify the fermion interactions with gauge and Higgs fields

Supersymmetry provides further unification (fermions in adj reps)

Forgacs + Manton ; Manton

Chapline + Slansky

Kubyshin + Mourao + Rudolph + Volobujev – book

Kapetanakis + G. Z. – Phys. Rept

Manousselis + G. Z., Phys. Lett. B504, 122 (01); PLB518, 171 (01); JHEP03, 002 (02); JHEP11, 025 (04)

Further successes

(a) chiral fermions in 4 dims from vector-like reps in the higher dim thy.

(b) the metric can be deformed (in certain non-symmetric coset sp) and more than one scales can be introduced

(c) Wilson flux breaking can be used

ADD

- Softly broken susy chiral ths in 4 dims can result from a higher dimensional susy theory

Theory in D dims $\longrightarrow$ Thy in 4 dims

1. Compactification $\underset{x^M}{M^D} \longrightarrow \underset{x^\mu}{M^4} \times \underset{y^a}{B}$

B – a compact space

$\dim B = D - 4 = d$

2. Dimensional Reduction

Demand that $\mathcal{L}$ is independent of the extra y^a coordinates

- One way: Discard the field dependence on y^a coordinates
- An elegant way: Allow field dependence on y^a and employ a symmetry of the Lagrangian to compensate.

Obvious choice: Gauge Symmetry

Allow a non-trivial dependence on y^a, but impose the condition that a symmetry transformation by an element of the isometry group S of B is compensated by a gauge transformation.

$\Longrightarrow$ $\mathcal{L}$ independent of y^a just because is gauge invariant.

Integrate out extra coordinates

CSDR: $B = S/R$

$$S: \; Q_A = \{ \underset{R}{Q_i} \, , \; \underset{S/R}{Q_a} \}$$

$$[Q_i, Q_j] = f_{ij}^{\;k} Q_k \; , \quad [Q_i, Q_a] = f_{ia}^{\;b} Q_b$$

$$[Q_a, Q_b] = f_{ab}^{\;i} Q_i + f_{ab}^{\;c} Q_c$$

vanishes in symmetric S/R

Consider a Yang-Mills-Dirac thy in D dims based on group G defined on $M^D \longrightarrow M^4 \times S/R$, $D = 4+d$

$$g^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & -g^{ab} \end{pmatrix}, \quad \eta^{\mu\nu} = \mathrm{dig}(1,-1,-1,-1)$$

g^{ab} - coset space metric

$d = \dim S - \dim R$

$$A = \int d^4x\, d^dy \sqrt{-g} \left[-\frac{1}{4} \mathrm{Tr}(F_{MN} F_{K\Lambda}) g^{MK} g^{N\Lambda} + \frac{i}{2} \bar{\psi} \Gamma^M D_M \psi \right]$$

$$D_M = \partial_M - \theta_M - A_M, \quad \theta_M = \frac{1}{2} \theta_{MN\Lambda} \Sigma^{N\Lambda}$$

spin connection of M^D

ψ in rep F of G

We require that any transformation by an element of S acting on S/R is compensated by gauge transformations.

$$A_\mu(x,y) = g(s)\, A_\mu(x, s^{-1}y)\, g^{-1}(s)$$

$$A_\alpha(x,y) = g(s)\, J_\alpha^b A_b(x, s^{-1}y)\, g^{-1}(s) + g(s)\, \partial_\alpha\, g^{-1}(s)$$

$$\psi(x,y) = f(s)\, \underline{\Omega}\, \psi(x, s^{-1}y)\, f^{-1}(s)$$

g, f - gauge transformations in the ad_J, F of G corresponding to the s transf. of S acting on S/R

J_α^b - Jacobian for s

$\underline{\Omega}$ - Jacobian + local Lorentz rotation in tangent space

Above conditions imply constraints that D-dims fields should obey.

Solution of constraints $\begin{cases} \cdot\ \text{4-dim fields} \\ \cdot\ \text{Potential} \\ \cdot\ \text{Remaining gauge invariance} \end{cases}$

Taking into account all the constraints and integrating out the extra coordinates, we obtain in 4 dims

$$A = C \int d^4x \Big(-\frac{1}{4} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \sum_a \mathrm{Tr}(D_\mu \phi_a D^\mu \phi_a)$$

$$+ V(\phi) + \frac{i}{2} \bar{\psi} \Gamma^\mu D_\mu \psi - \frac{i}{2} \bar{\psi} \Gamma^a D_a \psi \Big)$$

Kinetic terms mass terms

$$D_\mu = \partial_\mu - A_\mu,\ D_a = \partial_a - \theta_a - \phi_a,\ \theta_a = \frac{1}{2} \theta_{abc} \Sigma^{bc}$$

C - volume of coset space   spin connection of coset space

$$V(\phi) = -\frac{1}{4} g^{ac} g^{bd} \mathrm{Tr}\Big\{ \big(f_{ab}^{\ \ C} \phi_C - [\phi_a, \phi_b] \big) \big(f_{cd}^{\ \ D} \phi_D - [\phi_c, \phi_d] \big) \Big\}$$

$A = 1, \dots, \dim S$, f - structure const. of S

Still $V(\phi)$ only formal since ϕ_a must satisfy $f_{ai}^{\ \ D} \phi_D - [\phi_a, \phi_i] = 0$

- The 4-dim gauge group

$$H = C_G(R_G)$$

i.e. $G \supset R_G \times H$

$\uparrow$ higher dim group $\qquad \uparrow$ 4-dim group

- - Scalar fields

$$S \supset R$$

$$\text{adj}\, S = \text{adj}\, R + v$$

$$G \supset R_G \times H$$

$$\text{adj}\, G = (\text{adj}\, R, 1) + (1, \text{adj}\, H) + \sum (r_i, h_i)$$

If $v = \sum s_i$

when $s_i = r_i \Longrightarrow h_i$ survives in 4 dims

• • • fermions

$$G \supset R_G \times H$$

$$F = \sum (t_i, h_i)$$

spinor of $SO(d)$ under R

$$\sigma_d = \sum \sigma_j$$

for every $t_i = \sigma_i \Rightarrow h_i$ survives in 4 dims

Possible to obtain a chiral theory in 4 dims even starting with Weyl (+ Majorana) fermions in vector-like reps of G in $D = 4n+2$ dims.

If D is even

$$\Gamma^{D+1}\psi_\pm = \pm\psi_\pm \qquad \text{Weyl condition}$$

$$\psi = \psi_+ \oplus \psi_- = \sigma_D + \sigma_D'$$

non-self conjugate spinors of $SO(1, D-1)$

The $\left(SU(2)\times SU(2)\right)\times SO(d)$ branching rule is

$$\sigma_D = (2, 1; \sigma_d) + (1, 2; \sigma_d')$$

$$\sigma_D' = (2, 1; \sigma_d') + (1, 2; \sigma_d)$$

Starting with Dirac fermions

equal number of left- and

right-handed reps of the

4-dim group H

Weyl condition selects either σ_D

or σ_D'

Weyl condition cannot be applied in odd dims. In that case

$$\sigma_D = (2, 1\,; \sigma_d) + (1, 2\,; \sigma_d)$$

where σ_d is the unique spinor of $SO(d)$

$\rightarrow$ equal number of left- and right-handed reps in 4 dims.

Most interesting case is when $D = 4n+2$ and we start with a vectorlike rep. In that case σ_d is non-self-conjugate and $\sigma_d' = \bar{\sigma}_d$

Then the decomposition of $\sigma_d, \bar{\sigma}_d$ of $SO(d)$ under R is

$$\sigma_d = \sum \sigma_k \quad , \quad \bar{\sigma}_d = \sum \bar{\sigma}_k$$

Then

$$G \supset R_G \times H$$

$$f = \sum_i (r_i, h_i)$$

vectorlike

either self-conjugate or have a partner $(\bar{r}_i, \bar{h}_i)$

Then according to the rule from 6_d we will obtain in 4 dims left-handed fermions $f_L = \sum h_k^L$

Since 6_d is non-self-conjugate, f_L is non-self-conjugate.

Similarly from $\bar{6}_d$ we will obtain the right-handed rep $f_R = \sum \bar{h}_k^R = \sum h_k^L$

But since f vectorlike, $\bar{h}_k^R \sim h_k^L$

i.e. H is chiral theory

We can still impose Majorana condition (Weyl and Majorana are compatible in 4n+2 dims) to eliminate the doubling of fermion spectrum. Majorana cond (reverses the sign of all int. qu. nos) forces f_R to be the charge conjugate of f_L.

If f complex → chiral theory just $\bar{h}^R_K$ is different from h^L_K

An easy case in calculating the potential and its minimization

If $G \supset S \Rightarrow H$ breaks to $K = C_G(S)$

$$\begin{array}{ccccl} G \supset & S & \times & K & \leftarrow \text{gauge group after spontaneous sym. breaking} \\ & \cup & & \cap & \\ G \supset & R & \times & H & \\ & & & \uparrow & \\ & & & \text{gauge group in 4 dims} & \end{array}$$

But

fermion masses

$$M^2 \psi = D_a D^a \psi - \frac{1}{4} R \psi - \frac{1}{2} \Sigma^{ab} \underset{\substack{\shortparallel \\ 0 \\ \text{if } S \subset G}}{F_{ab}} \psi > 0$$

comparable to the compactification scale

Supersymmetry breaking by dim reduction over symmetric C.S.

(e.g. SO_7/SO_6)

Consider $G = E_8$ in 10 dims with Weyl-Majorana fermions in the adjoint of E_8, i.e. a susy E_8

Embedding of $R = SO(6)$ in E_8 is suggested by the decomposition

$$E_8 \supset SO(6) \times SO(10)$$

$$248 = (15,1) + (1,45) + (6,10)$$
$$+ (4,16) + (\bar{4},\overline{16})$$

$$\text{adj } S = \text{adj } R + v$$
$$21 = 15 + 6 \Leftarrow \text{vector}$$

Spinor of $SO(6)$: 4

In 4 dims we obtain a gauge thy based on

$$H = C_{E_8}(SO(6)) = SO(10)$$

with scalars in 10

and fermions in 16

- Theorem: When S/R symmetric the potential necessarily leads to spont. breakdown of H.

- Moreover in this case we have

$$\begin{array}{ccc} E_8 \supset SO(7) & \times & SO(9) \\ \cup & & \cap \\ E_8 \supset SO(6) & \times & SO(10) \end{array}$$

$\Rightarrow$ Final gauge group after breaking

$$K = C_{E_8}(SO(7)) = SO(9)$$

CSDR over symmetric coset spaces breaks completely original supersymmetry

Soft Supersymmetry Breaking by CSDR over non-symmetric CS.

We have examined the dim. red of a supersymmetric E_8 over the 3 existing 6-dim CS: G_2/SU_3, $Sp(4)/(SU(2)\times U(1))_{non-max}$, $SU(3)/U(1)\times U(1)$

$\Rightarrow$ Softly Broken Supersymmetric Theories in 4 dims without any further assumption

Non-symmetric CS admit torsion and the two latter more than one radii

Consider supersymmetric E_8 in 10 dims and $S/R = G_2/SU(3)$

We use the decomposition

$$E_8 \supset SU(3) \times E_6$$

$$248 = (8,1) + (1,78) + (3,27) + (\bar{3},\overline{27})$$

and choose $R = SU(3)$

$$\mathrm{adj}\, S = \mathrm{adj}\, R + v$$

$$14 = 8 + \underbrace{3+\bar{3}}_{\text{vector}}$$

Spinor : $1+3$ under $R = SU(3)$

$\Rightarrow$ 4 dim thy : $H = C_{E_8}(SU(3)) = E_6$

with scalars in $27 = 6$

and fermions in $27, 78$

i.e. spectrum of a supersymmetric E_6 thy in 4 dims

The Higgs potential of the genuine Higgs $\mathcal{B}$

$$V(\mathcal{B}) = 8 - \frac{40}{3}\mathcal{B}^2 - \left[4 d_{ijk}\mathcal{B}^i\mathcal{B}^j\mathcal{B}^k + h.c.\right]$$
$$+ \mathcal{B}^i\mathcal{B}^j d_{ijk} d^{klm} \mathcal{B}_l \mathcal{B}_m$$
$$+ \frac{11}{4}\sum_\alpha \mathcal{B}^i (G^\alpha)_i^j \mathcal{B}_j \mathcal{B}^k (G^\alpha)_k^l \mathcal{B}_l$$

which obtains F-terms contributions from the superpotential

$$W(\mathcal{B}) = \frac{1}{3} d_{ijk} \mathcal{B}^i \mathcal{B}^j \mathcal{B}^k$$

D-term contributions

$$\frac{1}{2} D^\alpha D^\alpha \quad , \quad D^\alpha = \sqrt{\frac{11}{2}}\, \mathcal{B}^i (G^\alpha)_i^j \mathcal{B}_j$$

The rest terms belong to the SSB part of the Lagrangian

$$\mathcal{L}_{scalar\,SSB} = -\frac{1}{R^2}\frac{140}{3}\mathcal{B}^2 - \left[4 d_{ijk}\mathcal{B}^i\mathcal{B}^j\mathcal{B}^k + h.c\right]\frac{9}{R}$$

$$M_{gaugino} = (1 + 3\tau)\frac{6}{\sqrt{3}}\frac{1}{R}$$

Reduction of 10-dim, $N=1$, E_8 over $S/R = SU(3)/U(1)\times U(1)$

N. Irges, G.Z.

We use the decomposition

$$E_8 \supset E_6 \times SU(3) \supset E_6 \times U(1)_A \times U(1)_B$$

and choose $R = U(1)_A \times U(1)_B$,

$$\Rightarrow H = C_{E_8}(U(1)_A \times U(1)_B) = E_6 \times U(1)_A \times U(1)_B$$

$$E_8 \supset E_6 \times U(1)_A \times U(1)_B$$

$$\begin{aligned} 248 = {} & 1_{(0,0)} + 1_{(0,0)} + 1_{(3,1/2)} + 1_{(-3,1/2)} \\ & + 1_{(0,-1)} + 1_{(0,1)} + 1_{(-3,-1/2)} + 1_{(3,-1/2)} \\ & + 78_{(0,0)} + 27_{(3,1/2)} + 27_{(-3,1/2)} + 27_{(0,-1)} \\ & + \overline{27}_{(-3,-1/2)} + \overline{27}_{(3,-1/2)} + \overline{27}_{(0,1)} \end{aligned}$$

$$\mathrm{adj}\, S = \mathrm{adj}\, R + v \leftarrow \text{vector}$$

$$\Downarrow$$

$$\begin{aligned} 8 = {} & (0,0) + (0,0) + (3,1/2) + (-3,1/2) \\ & + (0,-1) + (0,1) + (-3,-1/2) + (3,-1/2) \end{aligned}$$

$$SO(6) \supset SU(3) \supset U(1)_A \times U(1)_B$$

$$4 \quad = 1 + 3 = (0,0) + (3, 1/2) + (-3, 1/2) + (0, -1)$$

spinor

$\rightsquigarrow$ 4-dim theory

$$N = 1, \ E_6 \times U(1)_A \times U(1)_B$$

with chiral supermultiplets

$$A^i : 27_{(3,1/2)}, \ B^i : 27_{(-3,1/2)}, \ C^i : 27_{(0,-1)}$$

$$A : 1_{(3,1/2)}, \ B : 1_{(-3,1/2)}, \ C : 1_{(0,-1)}$$

Superpotential,

$$W(A^i, B^j, C^k, A, B, C) = \sqrt{40}\, d_{ijk} A^i B^j C^k + \sqrt{40}\, ABC$$

E_6 symmetric tensor

D-terms, $\frac{1}{2} D^\alpha D^\alpha + \frac{1}{2} D_1 D_1 + \frac{1}{2} D_2 D_2$

where $D^\alpha = \frac{1}{\sqrt{3}} \left(\alpha^i (G^\alpha)_i^j \alpha_j + b^i (G^\alpha)_i^j b_j + \gamma^i (G^\alpha)_i^j \gamma_j \right)$

$$D_1 = \sqrt{10/3} \left(\alpha^i (3\delta_i^j) \alpha_j + \bar{\alpha}(3)\alpha + b^i(-3\delta_i^j) b_j + \bar{b}(-3) b \right)$$

$$D_2 = \sqrt{\frac{40}{3}} \left(\alpha^i (\tfrac{1}{2}\delta_i^j) \alpha_j + \bar{\alpha}(\tfrac{1}{2})\alpha + b^i(\tfrac{1}{2}\delta_i^j) b_j + \bar{b}(\tfrac{1}{2}) b + \gamma^i(-1\delta_i^j)\gamma_j + \bar{\gamma}(-1)\gamma \right)$$

Soft scalar supersymmetry breaking terms, $\mathcal{L}_{scalar\,SSB}$

$$\mathcal{L}_{SSB} = \left(\frac{4R_1^2}{R_2^2R_3^2} - \frac{8}{R_1^2}\right)\alpha^i\alpha_i + \left(\frac{4R_1^2}{R_2^2R_3^2} - \frac{8}{R_1^2}\right)\bar{\alpha}\alpha$$

$$+ \left(\frac{4R_2^2}{R_1^2R_3^2} - \frac{8}{R_2^2}\right)b^ib_i + \left(\frac{4R_2^2}{R_1^2R_3^2} - \frac{8}{R_2^2}\right)\bar{b}b$$

$$+ \left(\frac{4R_3^2}{R_1^2R_2^2} - \frac{8}{R_3^2}\right)\gamma^i\gamma_i + \left(\frac{4R_3^2}{R_1^2R_2^2} - \frac{8}{R_3^2}\right)\bar{\gamma}\gamma$$

$$+ \left[\sqrt{280}\left(\frac{R_1}{R_2R_3} + \frac{R_2}{R_1R_3} + \frac{R_3}{R_2R_1}\right)d_{ijk}\,\alpha^i b^j\gamma^k\right.$$

$$\left.+ \sqrt{280}\left(\frac{R_1}{R_2R_3} + \frac{R_2}{R_1R_3} + \frac{R_3}{R_2R_1}\right)\alpha b\gamma + \text{h.c.}\right]$$

where $\alpha^i, b^i, \gamma^i, \alpha, b, \gamma$ are the scalar components of the A^i, B^i, C^i, A, B, C

Gaugino mass, $M = (1+3\tau)\dfrac{R_1^2+R_2^2+R_3^2}{8\sqrt{R_1^2R_2^2R_3^2}}$

torsion coef.

Potential, $V = V_F + V_D + V_{soft}$

The Wilson flux breaking

$M^4 \times B_0 \rightarrow M^4 \times B$, $B = B_0 / F^{S/R}$

$F^{S/R}$ – a freely acting discrete symmetry of B_0

1. B becomes multiply connected
2. For every element $g \in F^{S/R}$,

$\rightsquigarrow$ $U_g = P\exp\left(-i \int_{\gamma_g} T^a A^a_M(x)\, dx^M\right) \in H$

3. If the contour is non-contractible

$\rightsquigarrow$ $U_g \neq 1$ and then $f(g(x)) = U_g f(x)$ which leads to a breaking of H to $K' = C_H(T^H)$, where T^H is the image of the homomorphism of $F^{S/R}$ into H.

4. Matter fields invariant under $F^{S/R} \oplus T^H$

In the case of $SU(3)/U(1)\times U(1)$ a freely acting discrete group is

$$F^{S/R} = Z_3 \subset W, \quad W = \frac{W_S}{W_R}$$

$W_{S,R}$: Weyl group of S, R

$$\rightsquigarrow \gamma_3 = \mathrm{diag}(\mathbb{1}, \omega\mathbb{1}, \omega^2\mathbb{1}), \quad \omega = e^{2i\pi/3} \in Z_3$$

The fields that are invariant under $F^{S/R} \oplus T^H$ survive, i.e.

$$A_\mu = \gamma_3 A_\mu \gamma_3^{-1}$$

$$A^i = \omega\gamma_3 A^i, \; B^i = \omega^2\gamma_3 B^i, \; C^i = \omega^3\gamma_3 C^i$$

$$A = \omega A, \quad B = \omega^2 B, \quad C = \omega^3 C$$

$$\rightsquigarrow N=1, \quad SU(3)_c \times SU(3)_L \times SU(3)_R$$

with matter superfields in

$$(\overline{3}, 1, 3)_{(3,1/2)}, \; (3, \overline{3}, 1)_{(0,-1)}, \; (1, 3, \overline{3})_{(-3,1/2)}$$

$$\begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix} = q^c, \quad q = \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix}, \quad \lambda = \begin{pmatrix} N & E^c & \nu \\ E & N^c & e \\ \nu^c & e^c & S \end{pmatrix}$$

- Introducing non-trivial windings in R can appear 3 identical flavours in each of the bifundamental matter superfields.

Supersymmetry and gauge symmetry breaking

Consider the vevs in the scalars of $\lambda^{(1)}, \lambda^{(2)}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & V \end{pmatrix} , \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V & 0 & 0 \end{pmatrix}$$

$$\lambda^{(1)}: SU(3)_L \times SU(3)_R \times U(1)_A \times U(1)_B \Rightarrow SU(2)_L \times SU(2)_R \times U(1)$$

$$\lambda^{(2)}: SU(3)_L \times SU(3)_R \times U(1)_A \times U(1)_B \rightarrow SU(2)_L \times SU(2)_R \times U(1)'$$

their combination gives

$$SU(3)_L \times SU(3)_R \times U(1)_A \times U(1)_B \longrightarrow SU(2)_L \times U(1)_Y$$

electroweak breaking proceeds by

$$\begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & V \end{pmatrix}$$

For a certain relation among vevs the potential vanishes at the min.

Note that before EW breaking, supersymmetry is broken by D and F-terms, in addition to its breaking by soft terms.

- there is no proton decay
- the Froggatt-Nielsen mechanism is naturally realised.

Fuzzy CSDR

Aschieri
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$$M^D = M^4 \times (S/R)_F$$

finite matrix manifold
e.g. fuzzy sphere S^2_F

Instead of considering the algebra of functions

$$\mathrm{Fun}(M^D) \sim \mathrm{Fun}(M^4) \times \mathrm{Fun}(S/R)$$

we consider the algebra

$$A = \mathrm{Fun}(M^4) \times M_N$$

M_N - finite dim NC (non-com) algebra of matrices that approximates the functions on $(S/R)_F$

On A we still have the action of symmetry group S $\rightarrow$ we can apply CSDR

Fuzzy Sphere

Madore

Nice example of $(S/R)_F$ is the fuzzy sphere S^2_F, a matrix approximation of S^2. The algebra of functions on S^2 (spanned by spherical harmonics) is truncated at a given angular momentum and becomes finite dimensional. The algebra becomes that of $N \times N$ matrices.

The associativity of the algebra is nicely achieved by relaxing commutativity.

The algebra of functions on S^2 can be generated by the coordinates of R^3 modulo the relation $\sum_{a=1}^{3} x_a^2 = r^2$

Scalar functions on S^2 can be expanded

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}\, Y_{lm}(\theta,\phi)$$

spherical harmonics

$Y_{lm}(\theta,\phi)$ can be expressed in terms of the cartesian coordinates X_a, $a=1,2,3$ in $\mathbb{R}^3$

$$Y_{lm}(\theta,\phi)=\sum_a f^{(lm)}_{a_1\dots a_l}\, x^{a_1}\dots x^{a_l}$$

traceless symmetric tensor of $SO(3)$ with rank l

Similarly we can expand $N\times N$ matrices of a matrix theory on a fuzzy sphere

$$\hat{f}=\sum_{l=0}^{N-1}\sum_{m=-l}^{l} f_{lm}\,\hat{Y}_{lm}$$

$$\hat{Y}_{lm}=r^{-l}\sum_a f^{(lm)}_{a_1\dots a_l}\,\hat{x}^{a_1}\dots\hat{x}^{a_l}$$

where $f^{(lm)}_{a_1\dots a_l}$ the same as in S^2, while

$$\hat{x}_a=r\frac{i}{\sqrt{N^2-1}}X_a\ ,\qquad \hat{x}_a^{+}=\hat{x}_a$$

are $N\times N$ hermitian matrices proportional to the N-dim rep of the $SU(2)$ generators

They satisfy

$$\sum_{a=1}^{3} \hat{x}_a \hat{x}_a = r^2 \; , \quad [X_a, X_b] = \epsilon_{abc} X_c$$

$\hat{Y}_{lm}$ - fuzzy spherical harmonics

they obey $\; Tr_N \left(\hat{Y}^+_{lm} \hat{Y}_{l'm'} \right) = \delta_{ll'} \delta_{mm'}$

Obvious relation

$$\hat{f} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} f_{lm} \hat{Y}_{lm} \longrightarrow \sum_{l=0}^{N-1} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\theta, \phi)$$

Similarly

$$\frac{1}{N} Tr_N \longrightarrow \frac{1}{4\pi} \int d\Omega \; , \quad d\Omega = \sin\theta d\theta d\phi$$

In addition on S^2_F there is a natural $SU(2)$ covariant differential calculus. The derivations of a function f along X_a are given by

$$e_a(f) = [X_a, f] \; , \quad a = 1, 2, 3$$

i.e. this calculus is 3-dimensional.

These are essentially the angular momentum operators

$$J_a f = i e_a f = [i X_a, f]$$

which satisfy the $SU(2)$ Lie algebra

$$[J_a, J_b] = i \epsilon_{abc} J_c$$

In the limit $N \longrightarrow \infty$ the e_a become

$$e_a = \epsilon_{abc} x_b \partial_c$$

i.e. 2-dimensional

The exterior derivative is given by

$$df = [X_a, f] \theta^a$$

θ^a – 1-forms dual to e_a, $\langle e_a, \theta^b \rangle = \delta_a^b$

1-forms are generated by θ^a

$$\omega = \sum_{a=1}^{3} \omega_a \theta^a, \quad \omega \text{ any 1-form}$$

1-form on $M^4 \times S_F^2$: $A = A_\mu dx^\mu + A_a \theta^a$

with $A_\mu = A_\mu(x^\mu, x_a)$, $A_a = A_a(x^\mu, x_a)$

Non Commutative gauge fields and transformations

Consider a field $\phi(X_a)$ on a fuzzy space described by non-com coordinates X_a. An infinitesimal gauge transformation

$$\delta\phi(X_a) = \lambda(X_a)\,\phi(X_a)$$

$\lambda(X_a)$ - gauge tranformation parameter

$U(1)$ if $\lambda(X_a)$ antihermitian function of X_a

$U(P)$ if $\lambda(X_a)$ is valued in Lie algebra of $P \times P$ matrices

Coordinates X_a invariant under gauge transformation $\qquad \delta X_a = 0$

- $\delta(X_a \phi) = X_a \lambda(X_a)\phi \neq \lambda(X_a) X_a \phi$

- $\delta(\phi_a \phi) = \lambda(X_a)\phi_a \phi$

covariant coordinates

which holds if $\delta(\phi_a) = [\lambda(X_a), \phi_a]$

- $\phi_a = X_a + A_a$

NC analogue of covariant derivative

interpreted as gauge fields

note that $\delta A_a = -[X_a, \lambda] + [\lambda, A_a]$

supporting the interpretation of A_a

Correspondingly define

- $F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - C^c{}_{ab} A_c$

$= [\phi_a, \phi_b] - C^c{}_{ab}\phi_c$ analogue of field strength

$\rightsquigarrow \delta F_{ab} = [\lambda, F_{ab}]$

- $\delta\psi = [\lambda, \psi]$, spinor ψ in the adjoint

Actions in higher dimensions seen as 4-dim actions (expansion in Kaluza-Klein modes)

$$G = U(P) \quad \text{on} \quad M^4 \times (S/R)_F$$

$$A_{YM} = \frac{1}{4} \int d^4x \, Tr \, tr_G \, F_{MN} F^{MN}$$

integration over $(S/R)_F$

$$F_{MN} \longrightarrow (F_{\mu\nu}, F_{\mu b}, F_{ab})$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$$F_{\mu a} = \partial_\mu A_a - [X_a, A_\mu] + [A_\mu, A_a]$$

$$= \partial_\mu \phi_a + [A_\mu, \phi_a] = D_\mu \phi_a$$

$$F_{ab} = [\phi_a, \phi_b] - C^c{}_{ab} \phi_c$$

$$\rightsquigarrow A_{YM} = \int d^4x \, Tr \, tr_G \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu \phi_a)^2 - V(\phi) \right)$$

$$V(\phi) = -\frac{1}{4} Tr \, tr_G \sum_{ab} F_{ab} F_{ab}$$

$$= -\frac{1}{4} Tr \, tr_G \sum_{ab} \left([\phi_a, \phi_b] - C^c{}_{ab} \phi_c \right) \left([\phi_a, \phi_b] - C^c{}_{ab} \phi_c \right)$$

The infinitesimal G gauge transf

with parameter $\lambda(x^\mu, X^a)$ can be

interpreted as M^4 gauge transformation

$$\lambda(x^\mu, X^a) = \lambda^\alpha(x^\mu, X^a)\, \mathcal{T}^\alpha$$
$$= \lambda^{h,\alpha}(x^\mu)\, T^h\, \mathcal{T}^\alpha$$

$\mathcal{T}^\alpha$ - generators of $U(P)$

$\lambda^\alpha(x^\mu, X^a)$ - $N \times N$ matrices, therefore

expressible as

Kaluza-Klein modes of $\lambda(x^\mu, X^a)^\alpha$ $\Big\}$ $-\lambda(x^\mu)^{\alpha,h}\, T^h$

generators of $U(N)$

Considering on equal footing the indices h and α we interpret $\lambda^{h,\alpha}(x^\mu)$ as a field valued in the tensor product $\mathrm{Lie}(U(N)) \otimes \mathrm{Lie}(U(P)) = \mathrm{Lie}(U(NP))$

Similarly we write the gauge field A_ν as

$$A_\nu(x^\mu, X^a) = A_\nu^\alpha(x^\mu, X^a)\, \Upsilon^\alpha$$
$$= A_\nu^{h,\alpha}(x^\mu)\, T^h\, \Upsilon^\alpha$$

and interpret it as $\mathrm{Lie}(U(NP))$ valued gauge field on M^4.

Similarly for ϕ_a

Then we reduce the number of gauge fields and scalars by applying the CSDR principle.

e.g. $G = U(1)$, $(S/R)_F = S_F^2$

CSDR constraints are satisfied by embedding $SU(2)$ in $U(N)$.

We find in four dimensions

- No H group (due to the fact that the differential calculus is based on $\dim S$ derivations instead of $\dim S - \dim R$ in ordinary case)
- $K = C_{U(N)}(SU(2)) = U(N-2) \times U(1)$ as the final gauge group
- a harmless (singlet) surviving Higgs

Similar results are obtained for $G = U(P)$

CSDR for more general $(S/R)_F$

(e.g. CP^M described by $N\times N$ matrices)

CSDR constraints are satisfied by

embedding S in $U(N\cdot P)$

and the 4-dim gauge group is

$$K = C_{U(N\cdot P)}(S)$$

Concerning fermions, to solve the

corresponding constraints we embed

$$S \subset SO(\dim S)$$

$$U(N\cdot P) \supset S_{U(N\cdot P)} \times K$$

$$\operatorname{adj} U(N\cdot P) = (\operatorname{adj} S, 1) + (1, \operatorname{adj} K) + \sum_i (s_i, k_i)$$

$$SO(\dim S) \supset S$$

spinor $\sigma = \sum_i \sigma_i$

for $s_i = \sigma_i \rightsquigarrow k_i$ survive in 4 dims

Major difference among ordinary and fuzzy-CSDR

- 4-dim gauge theory appears already spontaneously broken

⇝ in 4 dims appears only the physical Higgs that survives SSB

⇝ Yukawa sector

(i) massive fermions

(ii) interactions among fermions and physical Higgs fields.

⇒ if we obtain in fuzzy-CSDR the SM ⇝ large extra dims

Fundamental differences among ordinary and fuzzy-CSDR:

- A non-abelian gauge group is not necessary in high dims.

The presence of a $U(1)$ in the higher-dim theory is enough to obtain non-abelian gauge theories in 4 dims.

- • The theory is renormalizable in the sense that divergencies can be removed by a finite number of counterterms.

Aschieri
Grammatikopoulos
Steinacker
Z
hep-th/0606021
JHEP
hep-th/07060398

We have constructed a renormalizable 4-dim $SU(N)$ gauge theory with suitable multiplet of scalar fields.

The symmetry breaking pattern and low-energy gauge group are determined dynamically in terms of a few free parameters of the potential. Depending on these parameters the final gauge group can be

$$SU(n) \quad \text{or} \quad SU(n_1) \times SU(n_2) \times U(1)$$

We explicitly found the tower of massive K-K modes, consistent with an interpretation as dimensionally reduced higher-dim gauge theory over an S_F^2.

The minima of the potential where vevs of scalars, $\langle \phi_a \rangle$ form the coordinates (generators) of a NC manifold (e.g. S^2_F, CP^N_F ...)

$\rightsquigarrow$ interpreted as spontaneously generated fuzzy extra dims.

Fluctuations around the vacuum: internal components of a higher-dim gauge field

$$\underbrace{\phi_a}_{\substack{\text{covariant}\\\text{coordinates}}} = \underbrace{\langle \phi_a \rangle}_{\text{coordinates}} + \underbrace{A_a}_{\substack{\text{gauge}\\\text{fields}}}$$

with a finite K-K tower of massive states.

Intermediate scales

$\rightsquigarrow$ Gauge theory on $M_4 \times M_{fuzzy}$

Low energy physics governed by zero modes

At high scales the theory behaves again as a 4-dim gauge theory maintaining renormalizability.

$\Longrightarrow$ Main features of dim red are realized within the framework of renormalizable 4-dim gauge thy.

Potential problem with chirality: In the best case only models with mirror fermions (not excluded exp)

Steinacker, Z '07
Chatzistavrakidis, Steinacker, Z '09

Chiral models demand additional requirements, e.g. orbifolding

Nice example

$SU(N)^3$ chiral models leading after further spontaneous breakings to $SU(3)^3$ and MSSM.

Chatzi/dis, Steinaker, Z
'10, '11

$N=4$ SYM

Particle content in $N=1$ language

- a $SU(3N)$ vector supermultiplet
- three adjoint chiral superplets Φ^i

and in components: $SU(3N)$ gauge bosons A_μ; 6 adjoint real scalars; ϕ_a (or 3 complex); 4 adjoint Majorana fermions

The theory has a global R-symmetry, $SU(4)_R$ under which the fieds transform:

- gauge fields as singlets
- real scalars as 6
- fermions as 4

Orbifolding by Z_3 embeded in $SU(3)$ as

$$SU(4)_R \supset SU(3) \times U(1)_R$$

$$\begin{aligned} 6 &= 3_2 + \bar{3}_{-2} \\ 4 &= 1_3 + 3_{-1} \end{aligned}$$

leads to $N=1$ theory. Kachru, Silverstein '98

Z_3 acts non-trivially on the various fields depending on their reps under the R-symmetry and the gauge group.

Orbifold projection keeps the fields which are invariant under the combined $\mathbb{Z}_3$ action (see e.g. Bailin + Love Phys. Rept '99)

The projected theory is

$N=1$, $SU(N)^3$ gauge thy

with chiral superfields in

$$3\left((N,\bar{N},1)+(\bar{N},1,N)+(1,N,\bar{N})\right)$$

i.e. chiral theory!

with 3 families!!!

However the $N=4$ superpotential,

$$W_{N=4} = \mathrm{Tr}\left(\epsilon_{ijk}\,\Phi^i\Phi^j\Phi^k\right)$$

is projected and gives the scalar pot,

$$V_{N=1}(\phi) = \frac{1}{4}\mathrm{Tr}\left(\left[\phi^i,\phi^j\right]^+\left[\phi_i,\phi_j\right]\right)$$

with minimum for vanishing vevs

$\rightsquigarrow$ No vacuum of NC-type!

Natural mechanism, aim for

- fuzzy vacua
- (potentially) realistic theory

require introduction of $N=1$ Soft Supersymmetry Breaking (SSB) terms, i.e. those that explicitly break $N=1$, but do not introduce quadratic divergences (Girandello-Grisaru '81): scalar mass terms, trilinear scalar interaction, gaugino masses.

$\rightsquigarrow$ Full potential is

$$V = V_{N=1} + V_{SSB} + V_D$$

D-terms ≥ 0

and can brought in the form

$$V = \frac{1}{4}\left(F^{ij}\right)^{\dagger} F^{ij} + V_D,$$

with $F^{ij} = [\phi^i, \phi^j] - i\epsilon^{ijk}\phi^{k\dagger}$

Vacuum

The minimum is obtained when

$$[\phi^i, \phi^j] = i\,\epsilon^{ijk}\,\phi^{k\dagger}$$
$$\phi^i \phi^{i\dagger} = R^2$$

compatible with Z_3 projection

Defining $\phi^i = \underline{\omega}\,\tilde{\phi}^i$

with $\underline{\omega} \neq 1$, $\underline{\omega}^3 = 1$, $\underline{\omega}^\dagger = \underline{\omega}^{-1}$;

$\tilde{\phi}^{i\dagger} = \tilde{\phi}^i$, i.e. $\phi^{i\dagger} = \underline{\omega}\,\phi^i$

$\rightsquigarrow [\tilde{\phi}^i, \tilde{\phi}^j] = i\,\epsilon^{ijk}\,\tilde{\phi}^k$; $\tilde{\phi}^i \tilde{\phi}^i = R^2$

i.e. ordinary fuzzy sphere.

The ϕ^is with fluctuations around the vacuum

$$\phi^i = \begin{pmatrix} \lambda^i_{(N-n)} + A^i & 0 & 0 \\ 0 & \omega\left(\lambda^i_{(N-n)} + A'^i\right) & 0 \\ 0 & 0 & \omega^2\left(\lambda^i_{(N-n)} + A''^i\right) \end{pmatrix}$$

with $\omega = {}^{2\pi}/_3$

The gauge symmetry $SU(N)^3$ is broken down to $SU(n)^3$

Moreover, there exist a finite K-K tower of massive states.

~ . ~ . ~ .

Particle Physics Models

Considering the embedding

$$SU(N) > SU(N-3) \times SU(3) \times U(1)$$

$$\rightsquigarrow SU(N) \longrightarrow SU(3)^3$$

$$SU(3)_c \times SU(3)_L \times SU(3)_R$$

$$3 \cdot \left((3, \bar{3}, 1) + (\bar{3}, 1, 3) + (1, 3, \bar{3}) \right)$$

~ . ~ . ~

Embedding in Matrix Models

$\rightsquigarrow$ Z_3-Orbifold Matrix M. Aoki, Iso, Suyama '02

$$Z_3 \subset SU(3) \times U(1) \subset SO(6) \subset SO(9,1)$$