Invariant Solutions of the Wheeler-DeWitt equation in Hybrid Gravity

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Field equations

The action of the hybrid metric-Palatini gravity is

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}[R + f(\mathcal{R})] + S_m, \tag{1}$$

where R is a metric Ricci curvature scalar and $f(\mathcal{R})$ is a function of the Palatini curvature scalar which is constructed by an independent torsionless connection $\hat{\Gamma}$. Here $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}(\hat{\Gamma})$.
The modified field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + f'(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{2}$$

The trace of (2) is called the hybrid structural equation. The Palatini curvature $\mathcal{R}$ can be expressed algebraically in terms of X, assuming that $f(\mathcal{R})$ has analytic solutions:

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa^2 T + R \equiv X. \tag{3}$$

Field equations

The action (1) is equivalent to the following one with the scalar field

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}[R+\phi\mathcal{R}-V(\phi)]. \tag{4}$$

where $\phi \equiv f'(\mathcal{R})$ and $V(\phi) = \mathcal{R}f'(\mathcal{R}) - f(\mathcal{R})$. Furthermore, for the two tensors $R_{\mu\nu}$ and $\mathcal{R}_{\mu\nu}$ it holds that[1]

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2}\frac{f(\mathcal{R})_{,\mu}f(\mathcal{R})_{,\nu}}{f^2(\mathcal{R})} - \frac{f(\mathcal{R})_{;\mu\nu}}{f(\mathcal{R})} - \frac{1}{2}\frac{\Box f(\mathcal{R})}{f(\mathcal{R})}g_{\mu\nu}$$

[1] S. Capozziello, T. Harko, T.S. Koivisto, F.S.N. Lobo, G.J Olmo, JCAP 04 (2013) 011 (arXiv:1209.2895)

Field equations

By using the last relation the action (4) becomes

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}[(1+\phi)R + \frac{3}{2\phi}\partial^\mu\phi\partial_\mu\phi - V(\phi)] \tag{5}$$

which is the action of a non minimally coupled scalar field.
For the FRW spatially flat spacetime and for empty space ($T_{\mu\nu} = 0$) the Lagrangian of the field equations is

$$\mathcal{L} = 6a\dot{a}^2(1+\phi) + 6a^2\dot{a}\dot{\phi} + \frac{3}{2\phi}a^3\dot{\phi}^2 + a^3V(\phi). \tag{6}$$

where the field equations are the Hamiltonian of (6) and the Euler-Lagrange equations with respect to the variables $x^i = (a, \phi)$.

Wheeler–DeWitt equation

The WDW equation which is a quantization of the Hamiltonian has a form

$$\Delta\Psi - a^3 V(\phi)\Psi = 0, \tag{7}$$

where Ψ is a wave function of the Universe and Δ is the Laplace operator

$$\Delta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^i}\right). \tag{8}$$

Point Symmetries

We consider a partial differential equation:

$$H\left(x^i, u, u_{,i}, u_{,ij}\right) = 0 \tag{9}$$

and a generator of an infinitesimal transformatin in the minisuperspace $\left\{x^i, u\right\}$ of the form:

$$X = \xi^i\left(x^j, u\right)\partial_i + \eta\left(x^j, u\right)\partial_u \tag{10}$$

We shall say that X is a Lie point symmetry of H if there exists a function λ, such that

$$X^{[2]}H = \lambda H\ ,\ \mathrm{mod}H = 0 \tag{11}$$

where $X^{[2]}$ is the second prologation of X.

Why do we need point symmetries?

In order to select potentials $V(\phi)$ in the case when the WDW equation admits Lie point symmetries, we will follow the geometric approach of
A. Paliathanasis, M. Tsamparlis, IJGMMP (2014) 14500376, (arXiv:1312.3942)
where Lie point symmetries are related to the conformal algebra of a minisuperspace.
As results, we will obtain forms of potentials with respective symmetries which are constructed of conformal Killing vector fields ξ_μ:

$$L_\xi g_{\alpha\beta} = 2\psi g_{\alpha\beta}, \quad \psi = \frac{1}{n}\xi^{\mu}_{;\mu}. \tag{12}$$

Results

- If $V(\phi) = V_0 \left(\sqrt{\phi} + V_1\right)^4$, the generic symmetry vector is

$$X_\Psi = c_1 \left(-\frac{1}{2}\partial_a + \frac{\phi + V_1\sqrt{\phi}}{a}\partial_\phi\right) + c_2 \Psi \partial_\Psi$$

$$f(\mathcal{R}) = \frac{\mathcal{R}^2}{4V_0} \quad \text{for} \quad V_1 = 0.$$

- If $V(\phi) = V_0 (1+\phi)^2 \exp\left(\frac{6}{V_1} \arctan\sqrt{\phi}\right)$, the generic symmetric vector is constructed of

$$X_1 = \partial_u, \tag{13}$$

$$X_\Psi = \Psi \partial_\Psi \tag{14}$$

$$X_2 = e^{-\frac{3v}{V_1}} \left(\left[\cos\left(V_C u - 3v\right)\right] \partial_u + \left[V_1 \cos\left(V_C u - 3v\right) + \sin\left(V_C u - 3v\right)\right] \partial_v \right) \tag{15}$$

$$X_3 = e^{-\frac{3v}{V_1}} \left(\left[\sin\left(V_C u + 3v\right)\right] \partial_u + \left[\cos\left(V_C u - 3v\right) - V_1 \sin\left(V_C u - 3v\right)\right] \partial_v \right) \tag{16}$$

Power law potential

Invariant solution

For the power law potential $V_0\left(\sqrt{\phi}+V_1\right)^4$ there exist a coordinate system $(a,\phi)\to(x,y)$ where the WDW becomes

$$\Psi_{,xx}+\Psi_{,yy}-2V_0y^4\Psi=0.$$

Thus by applying the zero order invariants of X_Ψ, $\{y, Ye^{\mu x}\}$ we can find the invariant solution

$$\Psi\left(x,y\right)=\sum_{\mu}\left[y_1e^{\mu x+w(y)}+y_2e^{\mu x-w(y)}\right] \tag{17}$$

where $w\left(y\right)=\frac{\sqrt{2}}{2}\int\sqrt{(2\bar{V}_0y^4-\mu^2)}dy$.

Lie point symmetries of the WDW eq. can be used in order to construct Noether point symmetries for a Lagrangian of field equations (see arXiv:1312.3942). We have the following results.

- If $V_1 = 0$, then $a(t) = a_0\sqrt{t}$, i.e. the radiation solution
- If $V_1 \neq 0$, then $a(\tau) = a_0(\tau - \tau_0) + a_1 - a_2\frac{1}{\tau - \tau_0}$, where $dt = a(\tau)\,d\tau$. However, if $a_0 = 0$ the Friedmann eq. H^2 can be written

$$\frac{H^2}{H_0^2} = \Omega_{0,r}a^{-4} + \Omega_{0,m}a^{-3} + \Omega_{0,k}a^{-2} + \Omega_{0,f}a^{-1} + \Omega_{0,\Lambda}$$

where $\Omega_{0,i} = \Omega_{0,i}(a_1)$, $i = \{r, m, k, f, \Lambda\}$ and $a_2 = \frac{(|a_1|+1)^2}{H_0}$.

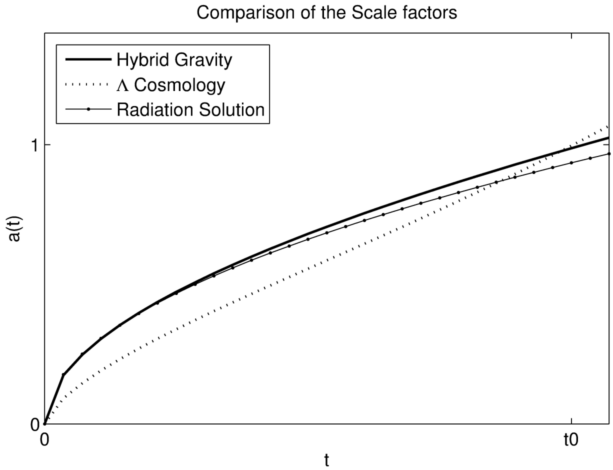
From the conformal transformation $dt = a(\tau)\,d\tau$ we have

$$\tau - \tau_0 = \exp\left[a_1 a_2^{-1}\tau_0 - a_2^{-1}t - W(w(t))\right] = (X(t))^{-1}, \tag{18}$$

where $w(t) = -a_1 a_2^{-1}\exp\left[a_2^{-1}(a_1\tau_0 - t)\right]$ and $W(t)$ is the Lambert W-function. One can obtain

$$a^2(t) = \left[a_2 X(t) - a_1\right]^2. \tag{19}$$

Comparison of the scale factors



Comparison of the scale factor (19) with that of ΛCDM-cosmology $a_\Lambda(t)$ and the radiation solution $a_r(t) = a_{0r}\sqrt{t}$ where t_0 is the present time, $a_\Lambda(t_0) = 1$. For the solution (19) of the Hybrid Gravity, we set $|a_1| > 1$.

Conclusion

- The Lie point symmetries of the WDW eq. in Hybrid Gravity were studied.
- The Lie invariants were used in order to find exact solution of the WDW and to solve analytically the modified field equations.
- It is of interest that in the case of the power law potential $V(\phi) = V_0 \left(\sqrt{\phi} + V_1\right)^4$ the Friedmann equation H^2 is a fourth order polynomial with non vanishing coefficients; that is, every power law term of $\sqrt{\phi}$ in the potential produces a corresponding fluid in the model.
$$\frac{H^2}{H_0^2} = \Omega_{0,r} a^{-4} + \Omega_{0,m} a^{-3} + \Omega_{0,k} a^{-2} + \Omega_{0,f} a^{-1} + \Omega_{0,\Lambda}$$

Thank you!