#### Killing-Yano tensors on toric Sasaki-Einstein spaces

Mihai Visinescu

Department of Theoretical Physics National Institute for Physics and Nuclear Engineering "Horia Hulubei" Bucharest, Romania

> 8-th Mathematical Physics Meeting Belgrade, August 24–31, 2014

#### Outline

- 1. Killing forms
- 2. Sasakian geometry
- 3. Symplectic approach
- 4. Delzant construction
- 5. Complex approach
- 6. Hidden symmetries on Sasaki-Einstein spaces
- 7. Example: Y(p, q) spaces
- 8. Outlook

#### References

- M. Visinescu, Mod. Phys. Lett. A 27, 1250217 (2012)
- M. Visinescu, G. E. Vîlcu, SIGMA 8, 058 (2012)
- M. Visinescu, J. Phys.: Conf. Series 411, 012030 (2013)
- M. Visinescu, J. Geom. Symm. Phys. 25, 93–104 (2013)
- A. M. Ionescu, V. Slesar, M. Visinescu, G. E. Vîlcu, *Rev. Math. Phys.* 25, 1330011 (2013)
- V. Slesar, M. Visinescu, G. E. Vîlcu, arXiv:1403.1015 [math.ph] (2014)

## Killing forms (1)

A conformal Killing-Yano tensor ( also called conformal Yano tensor or conformal Killing form or twistor form ) of rank p on a (pseudo-) Riemannian manifold (M, g) is a p -form  $\omega$  which satisfies [Yano, 1952]

$$abla_X \omega = rac{1}{p+1} X \lrcorner d\omega - rac{1}{n-p+1} X^* \wedge d^* \omega,$$

for any vector field X on M, where n is the dimension of M,  $X^*$  is the 1-form dual to the vector field X with respect to the metric g,  $\square$  is the operator dual to the wedge product and  $d^*$  is the adjoint of the exterior derivative d.

## Killing forms (2)

If  $\omega$  is co-closed then we obtain the definition of a Killing-Yano tensor, also called Yano tensor or Killing form :

 $\omega_{i_1\ldots i_{k-1}(i_k;j)}=\mathbf{0}\,.$ 

Here a semicolon precedes an index of covariant differentiation associated with the Levi-Civita connection and a round bracket denotes a symmetrization over the indices within.

Moreover, a Killing form  $\omega$  is said to be a **special Killing form** if it satisfies for some constant *c* the additional equation

$$\nabla_{\boldsymbol{X}}(\boldsymbol{d}\omega) = \boldsymbol{c}\boldsymbol{X}^* \wedge \omega\,,$$

for any vector field X on M.

There is also a symmetric generalization of the Killing vectors:

A symmetric tensor  $K_{i_1 \dots i_r}$  of rank r > 1 satisfying the generalized Killing equation

 $K_{(i_1\cdots i_r;j)}=0,$ 

is called a Stäckel-Killing tensor .

## Killing forms (4)

The conserved quantities associated with Killing tensors are given by the following proposition:

For any geodesic  $\gamma$  with tangent vector  $\dot{\gamma}^i$ 

$$Q_K = K_{i_1\cdots i_r} \dot{\gamma}^{i_1}\cdots \dot{\gamma}^{i_r},$$

is constant along  $\gamma$  .

Let us note that there is an important connection between these two generalizations of the Killing vectors. To wit, given two Killing-Yano tensors  $\psi^{i_1,...,i_k}$  and  $\sigma^{i_1,...,i_k}$  there is a Stäckel-Killing tensor of rank 2 :

$$\mathcal{K}_{ij}^{(\psi,\sigma)} = \psi_{ij_2\dots i_k} \sigma_j^{i_2\dots i_k} + \sigma_{ij_2\dots j_k} \psi_j^{i_2\dots i_k}.$$

This fact offers a method to generate higher order integrals of motion by identifying the complete set of Killing-Yano tensors.

# Sasakian geometry (1)

Contact structures(1)

A (2n + 1)-dimensional manifold *M* is a **contact manifold** if there exists a 1-form  $\eta$ , called a **contact 1-form**, on *M* such that

 $\eta \wedge (\boldsymbol{d}\eta)^n \neq \mathbf{0}$ .

For every choice of contact 1-form  $\eta$  there exists a unique vector field  $K_{\eta}$ , called the **Reeb vector field**, that satisfies

$$\eta(K_\eta) = 1$$
 and  $K_\eta ot d\eta = 0$ .

## Sasakian geometry (2)

**Contact structures(2)** 

**Example:** Consider  $\mathbb{R}^{2n+1}$  with Cartesian coordinates

$$(x^1,\ldots,x^n;y^1,\ldots,y^n;z).$$

Contact 1-form, Reeb vector field and Riemannian metric:

$$\eta = dz - \sum_{i}^{n} y^{i} dx^{i}$$
, $\mathcal{K}_{\eta} = rac{\partial}{\partial z}$ , $g = \eta^{2} + \sum_{i}^{n} \left( (dx^{i})^{2} + (dy^{i})^{2} 
ight)$ .

## Sasakian geometry (3)

A simple and direct definition of the Sasakian structures is the following:

A compact Riemannian manifold (Y, g) is **Sasakian** if and only if its metric cone  $(X = C(Y) \cong \mathbb{R}_+ \times Y, \ \overline{g} = dr^2 + r^2 g)$  is Kähler.

Here  $r \in (0, \infty)$  may be considered as a coordinate on the positive real line  $\mathbb{R}_+$ . The Sasakian manifold (Y, g) is naturally isometrically embedded into the metric cone via the inclusion  $Y = \{r = 1\} = \{1\} \times Y \subset C(Y)$ .

## Sasakian geometry (4)

Let us denote by

$$\tilde{K} \equiv \mathcal{J}\left(r\frac{\partial}{\partial r}\right),$$

where  $\mathcal{J}$  is the complex structure on the cone manifold.  $\tilde{K} - i\mathcal{J}\tilde{K}$  is a holomorphic vector field on C(Y) and the restriction K of  $\tilde{K}$  to  $Y \subset C(Y)$  is the **Reeb** vector field on Y. The Reeb vector field K is a Killing vector on (Y, g), has unit length and, in particular, is nowhere zero.

Let *Y* be a Sasaki-Einstein manifold of dimension  $\dim_{\mathbb{R}} Y = 2n - 1$  and its Kähler cone X = C(Y) is of dimension  $\dim_{\mathbb{R}} X = 2n$ ,  $(\dim_{\mathbb{C}} X = n)$ .

## Sasakian geometry (5)

Sasaki-Einstein geometry is naturally "sandwiched" between two Kähler-Einstein geometries as shown in the following proposition:

Let (Y, g) be a Sasaki manifold of dimension 2n - 1. Then the following are equivalent

- (1) (Y,g) is Sasaki-Einstein with  $Ric_g = 2(n-1)g$ ;
- (2) The Kähler cone  $(C(Y), \bar{g})$  is Ricci-flat,  $Ric_{\bar{g}} = 0$ ;
- (3) The transverse Kähler structure to the Reeb foliation  $\mathcal{F}_{\mathcal{K}}$  is Kähler-Einstein with  $Ric^{T} = 2ng^{T}$ .

## Sasakian geometry (6)

The Kähler form  $\omega$  is an exact 2-form and homogeneous degree 2 under the Euler vector  $r\frac{\partial}{\partial r}$ 

$$\omega = -\frac{1}{2}d(r^2\eta) = -rdr \wedge \eta - \frac{1}{2}r^2d\eta \quad ,$$
  
 $\mathcal{L}_{rrac{\partial}{\partial r}}\omega = 2\omega,$ 

where  $\eta$  is the Sasakian 1-form of Y. It lifts to C(Y) as

$$\eta = \mathcal{J}\left(\frac{dr}{r}\right) = i(\partial - \bar{\partial})\log r = -2i\partial\bar{\partial}\log r.$$

## Sasakian geometry (7)

Note that the Reeb vector  $\tilde{K}$  is dual to the 1-form  $r^2\eta$ . The Kähler form  $\omega$  can be written as

$$\omega = \frac{1}{2}i\partial\bar{\partial}r^2,$$

which means that

$$F=rac{r^2}{4},$$

is the Kähler potential.

### Symplectic approach (1)

Let  $(y, \phi)$  be the symplectic coordinates on *X*. If  $(X, \omega)$  is toric, the standard *n*-torus  $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$  acts effectively on *X* 

 $\tau: \mathbb{T}^n \to Diff(X, \omega),$ 

preserving the Kähler form.  $\partial/\partial \phi_i$  generate the  $\mathbb{T}^n$  action,  $\phi_i$  being the angular coordinates along the orbit of the torus action  $\phi_i \sim \phi_i + 2\pi$ .  $\mathbb{T}^n$ -invariant Kähler metric on *X* is

$$ds^2 = G_{ij}dy_idy_j + G^{ij}d\phi_id\phi_j,$$

where  $G_{ij}$  is the Hessian of the symplectic potential G(y) in the *y* coordinates

$$G_{ij} = rac{\partial^2 G}{\partial y_i \partial y_j}$$
,  $1 \le i,j \le n,$ 

and  $G^{ij} = (G_{ij})^{-1}$ .

### Symplectic approach (2)

The almost complex structure is

$$\mathcal{J} = \left( \begin{array}{cc} 0 & -G^{ij} \\ G_{ij} & 0 \end{array} \right),$$

and the symplectic (Kähler) form is  $\omega = dy_i \wedge d\phi_i$ . Associated to  $(X, \omega, \tau)$  there is a moment map  $\mu : X \to \mathbb{R}^n$ 

 $\mu(\mathbf{y},\phi)=\mathbf{y},$ 

i.e. the projection on the action coordinates:

$$y_i = -\frac{1}{2} \left\langle r^2 \eta, \frac{\partial}{\partial \phi_i} \right\rangle.$$

## Symplectic approach (3)

Let us write the Reeb vector in the form:

$$\tilde{K} = b_i \frac{\partial}{\partial \phi_i}.$$

In the symplectic coordinates  $(y, \phi)$  we have

$$r\frac{\partial}{\partial r}=2y_i\frac{\partial}{\partial y_i},$$

and the components of the Reeb vector are  $b_i = 2G_{ij}y_j$ .

## Symplectic approach (4)

The moment map exhibits the Kähler cone as a Lagrangian fibration over a strictly convex rational polyhedral cone  $C \subset \mathbb{R}^n$  by forgetting the angular coordinates  $\phi_i$ 

$$\mathcal{C}\left\{y\in\mathbb{R}^{n}|I_{a}(y)\geqslant0$$
,  $a=1,\ldots,d
ight\},$ 

with the linear function  $l_a(y) = (y, v_a)$ , where  $v_a$  are the inward pointing normal vectors to the *d* facets of the polyhedral cone. The set of vectors  $\{v_a\}$ 

$$\mathbf{v}_{\mathbf{a}} = \mathbf{v}_{\mathbf{a}}^{i} \frac{\partial}{\partial \phi_{i}} \quad , \quad \mathbf{v}_{\mathbf{a}}^{i} \in \mathbb{Z},$$

is called a toric data.

#### **Delzant construction (1)**

The image of X under the moment map  $\mu$  is a certain kind of convex rational polytope in  $\mathbb{R}^n$  called Delzant polytope.

A convex polytope P in  $\mathbb{R}^n$  is **Delzant** [Delzant, 1988] if

(a) there are *n* edges meeting at each vertex *p*;

- (b) the edges meeting at the vertex *p* are rational, i.e. each edge is of the form  $1 + tu_i$ ,  $0 \le t \le \infty$  where  $u_i \in \mathbb{Z}^n$ ;
- (c) the  $u_i, \ldots, u_n$  in (b) can be chosen to be a basis of  $\mathbb{Z}^n$ .

Delzant construction associates to every Delzant polytope  $P \subset \mathbb{R}^n$  a closed connected symplectic manifold  $(M, \omega)$  together with the Hamiltonian  $\mathbb{T}^n$  action and the moment map  $\mu$ .

#### **Delzant construction (2)**

Using the Delzant construction the general symplectic potential has the following form in terms of the toric data:

 $G=G^{can}+G_b+h,$ 

where

$$G^{can} = \frac{1}{2} \sum_{a} l_a(y) \log l_a(y),$$
$$G_b = \frac{1}{2} \sum_{a} l_b(y) \log l_b(y) - \frac{1}{2} l_\infty(y) \log l_\infty(y),$$

with  $l_b(y) = (b, y)$ ,  $l_{\infty}(y) = \sum_a (v_a, y)$  and *h* is a homogeneous degree one function of variables  $y_i$ 

$$h = \lambda_i y_i + t,$$

 $\lambda_i$ , *t* being some constants.

## **Complex approach (1)**

The standard complex coordinates are  $w_i$  on  $\mathbb{C}\setminus\{0\}$ . Log complex coordinates are  $z_i = \log w_i = x_i + i\phi_i$  and in these complex coordinates the metric is

 $ds^2 = F_{ij}dx_idx_j + F_{ij}d\phi_id\phi_j,$ 

where  $F_{ij}$  is the Hessian of the Kähler potential. Note also that in the complex coordinates  $z_i$  the complex structures and the Kähler form are:

$$\mathcal{J} = \left( \begin{array}{cc} \mathbf{0} & -\mathbb{I} \\ \mathbb{I} & \mathbf{0} \end{array} \right) \quad , \quad \omega = \left( \begin{array}{cc} \mathbf{0} & F_{ij} \\ -F_{ij} & \mathbf{0} \end{array} \right).$$

#### **Complex approach (2)**

The symplectic potential G and Kähler potential F are related by the Legendre transform

$$F(x) = \left(y_i \frac{\partial G}{\partial y_i} - G\right) \ (y = \partial F / \partial x).$$

Therefore *F* and *G* are Legendre dual to each other

$$F(x) + G(y) = \sum_{j} \frac{\partial F}{\partial x_{j}} \frac{\partial G}{\partial y_{j}}$$
 at  $x_{i} = \frac{\partial G}{\partial y_{i}}$  or  $y_{i} = \frac{\partial F}{\partial x_{i}}$ .

It follows from that  $F_{ij} = G^{ij}$   $(y = \partial F / \partial x)$ .

#### **Complex approach (3)**

The (n, 0) holomorphic form of the Ricci-flat metric on the Calabi-Yau cone is

$$dV = e^{i\alpha} (\det F_{ij})^{1/2} dz_1 \wedge \cdots \wedge dz_n,$$

with  $\alpha$  a phase space which is fixed by requiring that dV is a closed form. The complex coordinates can be chosen such that

$$dV = e^{x_1 + i\phi_1} dz_1 \wedge \cdots \wedge dz_n = dw_1 \wedge \cdots \wedge dw_n / (w_2 \dots w_n).$$

The Kähler potential *F* is obtain by the Legendre transform

$$F(x) = \frac{r^2}{4} = \frac{1}{2} \sum_{i} b_i y_i - t,$$

Detailed analysis shows that the constant *t* must be set to zero..

#### Hidden symmetries on Sasaki-Einstein spaces (1)

The Killing forms of the toric Sasaki-Einstein manifold Y are described by the special Killing forms

 $\Theta_k = \eta \wedge (d\eta)^k$ ,  $k = 0, 1, \cdots, n-1$ .

Besides these Killing forms, there are n - 1 closed conformal Killing forms (also called \*-Killing forms)

$$\Phi_k = (d\eta)^k \quad , \quad k = 1, \cdots, n-1$$

#### Hidden symmetries on Sasaki-Einstein spaces (2)

Moreover in the case of the Calabi-Yau cone, the holonomy is SU(n) and there are *two additional* Killing forms of degree *n*. In order to write explicitly these additional Killing forms we shall express the volume form of the metric cone in terms of the Kähler form

$$d\mathcal{V}=rac{1}{n!}\omega^n.$$

Here  $\omega^n$  is the wedge product of  $\omega$  with itself *n* times. The volume of a Kähler manifold can be also written as

$$d\mathcal{V}=\frac{i^n}{2^n}(-1)^{n(n-1)/2}dV\wedge\overline{dV},$$

where dV is the complex volume holomorphic (n, 0) form of C(Y). The additional (real) Killing forms are given by the real respectively the imaginary part of the complex volume form. [Semmelmann, 2003]

#### Hidden symmetries on Sasaki-Einstein spaces (3)

In order to extract the corresponding additional Killing forms of the Einstein-Sasaki spaces we make use of the fact that for any p-form  $\psi$  on the space Y we can define an associated p + 1-form  $\psi^{C}$  on the cone C(Y):

$$\psi^{\mathcal{C}} := r^{p} dr \wedge \psi + rac{r^{p+1}}{p+1} d\psi.$$

 $\psi^{C}$  is parallel if and only if  $\psi$  is a special Killing form with constant c = -(p+1).

## Y(p,q) spaces (1)

Infinite family Y(p, q) of Einstein-Sasaki metrics on  $S^2 \times S^3$ provides supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space Y(p, q) of an  $S^1$ -fibration over  $S^2 \times S^2$  with relative prime winding numbers p and q is topologically  $S^2 \times S^3$ .

Explicit local metric of the 5-dimensional Y(p, q) manifold given by the line element [Gauntlett, Martelli, Sparks, Waldram, 2004]

$$ds_{ES}^{2} = \frac{1-cy}{6} (d\theta^{2} + \sin^{2}\theta \, d\phi^{2}) + \frac{1}{w(y)q(y)} dy^{2} + \frac{q(y)}{9} (d\psi - \cos\theta \, d\phi)^{2} + w(y) \left[ d\alpha + \frac{ac - 2y + cy^{2}}{6(a - y^{2})} [d\psi - \cos\theta \, d\phi] \right]^{2},$$

where

## Y(p,q) spaces (2)

$$w(y) = \frac{2(a-y^2)}{1-cy}$$
,  $q(y) = \frac{a-3y^2+2cy^3}{a-y^2}$ 

and *a*, *c* are constants. The constant *c* can be rescaled by a diffeomorphism and in what follows we assume c = 1. The coordinate change  $\alpha = -\frac{1}{6}\beta - \frac{1}{6}c\psi'$ ,  $\psi = \psi'$  takes the line element to the following form ( with p(y) = w(y)q(y) )

$$ds_{ES}^2 = \frac{1-y}{6} (d\theta^2 + \sin^2\theta \, d\phi^2) + \frac{1}{p(y)} dy^2 + \frac{p(y)}{36} (d\beta + \cos\theta \, d\phi)^2 + \frac{1}{9} [d\psi' - \cos\theta \, d\phi + y (d\beta + \cos\theta \, d\phi)]^2,$$

## Y(p,q) spaces (3)

For

 $0 < \alpha < 1$ ,

we can take the range of the angular coordinates  $(\theta, \Phi, \Psi)$  to be  $0 \le \theta \le 2\pi, 0 \le \Phi \le 2\pi, 0 \le \Psi \le 2\pi$ . Choosing 0 < a < 1 the roots  $y_i$  of the cubic equation

 $a-3y^2+2y^3=0\,,$ 

are real, one negative  $(y_1)$  and two positive  $(y_2, y_3)$ . If the smallest of the positive roots is  $y_2$ , one can take the range of the coordinate y to be

 $y_1 \leq y \leq y_2.$ 

## Y(p,q) spaces (4)

For this particular space we take the complex coordinates

$$z^{1} = 3 \ln r + \ln \sin \theta + \frac{1}{2} \ln \frac{p(y)(1-y)}{2} + i\psi'$$

$$z^2 = \frac{1}{3\sqrt{3}} \left( \ln \tan \frac{\theta}{2} + \mathrm{i}\phi \right)$$

$$z^{3} = \frac{1}{6} \ln \left( \frac{1}{\sin \theta} \sqrt{(y - y_{1})^{-\frac{1}{y^{1}}} (y_{2} - y)^{-\frac{1}{y^{2}}} (y_{3} - y)^{-\frac{1}{y^{3}}}} \right)$$
$$- i\alpha - \frac{1}{6} i\psi'$$

## Y(p,q) spaces (5)

One can write

$$ds_{ES}^2 = ds_{EK}^2 + (\frac{1}{3}d\psi' + \sigma)^2$$

The Sasakian 1-form of the Y(p,q) space is

$$\eta = \frac{1}{3} d\psi' + \sigma \,,$$

with

$$\sigma = \frac{1}{3} [-\cos\theta \, d\phi + y(d\beta + \cos\theta \, d\phi)] \, .$$

connected with local Kähler form  $\Omega_{EK}$ .

This form of the metric with the 1-form  $\eta$  is the standard one for a locally Einstein-Sasaki metric with  $\frac{\partial}{\partial \psi'}$  the Reeb vector field.

## Y(p,q) spaces (6)

The local Kähler and holomorphic (2,0) form for  $ds_{FK}^2$  are

$$\Omega_{EK} = \frac{1-y}{6} \sin\theta d\theta \wedge d\phi + \frac{1}{6} dy \wedge (d\beta + \cos\theta d\phi)$$
$$dV_{EK} = \sqrt{\frac{1-y}{6p(y)}} (d\theta + i\sin\theta d\phi) \wedge \left[ dy + i\frac{p(y)}{6} (d\beta + \cos\theta d\phi) \right]$$

## Y(p,q) spaces (7)

From the isometries  $SU(2) \times U(1) \times U(1)$  the momenta  $P_{\phi}, P_{\psi}, P_{\alpha}$  and the Hamiltonian describing the geodesic motions are conserved.  $P_{\phi}$  is the third component of the SU(2) angular momentum, while  $P_{\psi}$  and  $P_{\alpha}$  are associated with the U(1) factors. Additionally, the total SU(2) angular momentum given by

$$J^2=P_ heta^2+rac{1}{\sin^2 heta}(P_\phi+\cos heta P_\psi)^2+P_\Psi^2\,,$$

is also conserved.

## Y(p,q) spaces (8)

Specific conserved quantities for Einstein-Sasaki spaces (1)

First of all from the 1-form  $\eta$ 

$$\begin{split} \Psi &= \eta \wedge d\eta \\ &= \frac{1}{9} [(1-y) \sin \theta \, d\theta \wedge d\phi \wedge d\psi' + dy \wedge d\beta \wedge d\psi' \\ &+ \cos \theta \, dy \wedge d\phi \wedge d\psi' - \cos \theta \, dy \wedge d\beta \wedge d\phi \\ &+ (1-y)y \sin \theta \, d\beta \wedge d\theta \wedge d\phi] \,. \end{split}$$

is a special Killing form. Let us note also that

$$\Psi_k = (d\eta)^k, \quad k = 1, 2,$$

are closed conformal Killing forms (\*-Killing forms).

## Y(p,q) spaces (9)

Specific conserved quantities for Einstein-Sasaki spaces (2)

On the Calabi-Yau manifold the Kähler form is

 $\Omega_{\textit{cone}} = \textit{rdr} \wedge \eta + \textit{r}^2 \Omega_{\textit{EK}}$  .

and the holomorphic (3,0) form is

$$dV_{cone} = e^{\psi'} r^2 dV_{EK} \wedge [dr + ir \wedge \eta]$$
  
=  $e^{\psi'} r^2 \sqrt{\frac{1-y}{6p(y)}} (d\theta + i\sin\theta d\phi)$   
 $\wedge \left[ dy + i \frac{p(y)}{6} (d\beta + \cos\theta d\phi) \right]$   
 $\wedge \left[ dr + i \frac{r}{3} [yd\beta + d\psi' - (1-y)\cos\theta d\phi] \right]$ 

## Y(p,q) spaces (10)

Specific conserved quantities for Einstein-Sasaki spaces (3) The additional Killing 3-forms of the Y(p, q) spaces are extracted from the volume form  $dV_{cone}$ .

Using the the 1-1 correspondence between special Killing p-forms on  $M_{2n+1}$  and parallel (p + 1)-forms on the metric cone  $C(M_{2n+1})$  for p = 2 we get the following additional Killing 2-forms of the Y(p, q) spaces written as real forms:

$$\Xi = \operatorname{Re} \omega^{M} = \sqrt{\frac{1-y}{6\,p(y)}} \\ \times \left(\cos\psi' \left[-dy \wedge d\theta + \frac{p(y)}{6}\sin\theta\,d\beta \wedge d\phi\right] \right. \\ \left. -\sin\psi' \left[-\sin\theta\,dy \wedge d\phi - \frac{p(y)}{6}d\beta \wedge d\theta \right. \\ \left. + \frac{p(y)}{6}\cos\theta\,d\theta \wedge d\phi\right] \right)$$

#### Y(p,q) spaces (11)

Specific conserved quantities for Einstein-Sasaki spaces (4)

$$\begin{split} \Upsilon &= \operatorname{Im} \omega^{M} = \sqrt{\frac{1-y}{6\,p(y)}} \\ &\times \left( \sin \psi' \Big[ -dy \wedge d\theta + \frac{p(y)}{6} \sin \theta \, d\beta \wedge d\phi \Big] \\ &+ \cos \psi' \Big[ -\sin \theta \, dy \wedge d\phi - \frac{p(y)}{6} d\beta \wedge d\theta \\ &+ \frac{p(y)}{6} \cos \theta \, d\theta \wedge d\phi \Big] \Big) \end{split}$$

## Y(p,q) spaces (12)

Specific conserved quantities for Einstein-Sasaki spaces (5)

The Stäckel-Killing tensors associated with the Killing forms  $\Psi, \Xi, \Upsilon$  are constructed as usual. Together with the Killing vectors  $P_{\phi}, P_{\psi}, P_{\alpha}$  and the total angular momentum  $J^2$  these Stäckel-Killing tensors provide the superintegrability of the Y(p, q) geometries.

#### Outlook

- Complete integrability of geodesic equations
- Separability of Hamilton-Jacobi, Klein-Gordon, Dirac equations
- Hidden symmetries of other spacetime structures