Exact and asymptotic solutions for cosmological models with non-minimally coupled scalar fields

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based on
M.A. Skugoreva, A.V. Toporensky, S.Yu. V., arXiv:1404.6226;
I.Ya. Aref'eva, N.V. Bulatov, R.V. Gorbachev, S.Yu. V.,
Class. Quant. Grav. **31** (2014) 065007, arXiv:1206.2801;
A.Yu. Kamenshchik, E.O. Pozdeeva, A. Tronconi, G. Venturi, S.Yu. V.,
Class. Quant. Grav. **31** (2014) 105003, arXiv:1312.3540

Models with scalar fields are very useful to describe the observable evolution of the Universe as the dynamics of the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) background with

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In my talk I present 3 examples how we can use our knowledge about models with minimally coupled scalar field working in the Jordan frame.

MODEL WITH NON-MINIMAL COUPLING

Models with non-minimally coupled scalar fields are described by the following action:

$$S = \int d^4x\sqrt{-g}\left[U(\phi)R - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right], \qquad (1)$$

where $U(\phi)$ and $V(\phi)$ are differentiable functions of the scalar field ϕ. We assume that $U(\phi) \geqslant 0$.

In the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric with the interval:

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

we get the following system of equations:

$$6UH^2 + 6\dot{U}H = \frac{1}{2}\dot{\phi}^2 + V\,, \tag{2}$$

$$2U\left(2\dot{H} + 3H^2\right) = -\frac{\dot{\phi}^2}{2} - 2\ddot{U} - 4H\dot{U} + V\,, \tag{3}$$

$$\ddot{\phi} + 3H\dot{\phi} - 6U'\left(\dot{H} + 2H^2\right) + V' = 0\,. \tag{4}$$

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From Eqs. (2)–(4) we get the following system:

$$\begin{aligned}
\dot{\phi} &= \psi, \\
\dot{\psi} &= -3H\psi - \frac{(6U''+1)U'}{2\,(3U'^2+U)}\psi^2 + \frac{UV' - 2VU'}{3U'^2+U}, \\
\dot{H} &= -\frac{2U''+1}{4(3U'^2+U)}\psi^2 + \frac{2U'H\psi}{3U'^2+U} - \frac{6U'^2H^2}{3U'^2+U} + \frac{U'V'}{2(3U'^2+U)}.
\end{aligned} \tag{5}$$

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\end{aligned} \tag{5}$$

Note that equation (2) is not a consequence of system (5).
The system (5) is equivalent to the initial system of equations (2)–(4) if and only if we choose such initial data that equation (2) is satisfied.
In other words, if equation (2) is satisfied in the initial moment of time, then from system (5) it follows that equation (2) is satisfied at any moment of time.

Subtracting (2) from Eq. (3), we obtain:

$$4U\dot{H} = -\dot{\phi}^2 - 2\ddot{U} + 2H\dot{U}. \tag{6}$$

We introduce a new variable

$$P \equiv \frac{H}{\sqrt{U}} + \frac{U'\dot{\phi}}{2U\sqrt{U}}.$$

In terms of P Eqs. (2) and (6) have the following form

$$3P^2 = \frac{U + 3U'^2}{4U^3}\dot{\phi}^2 + \frac{V}{2U^2} = \frac{A}{2}\dot{\phi}^2 + V_{eff}\,, \tag{7}$$

$$\dot{P} = -\frac{A\sqrt{U}}{2}\dot{\phi}^2, \tag{8}$$

where $A \equiv (U + 3U'^2)/(2U^3)$, $V_{eff} \equiv V/(2U^2)$.
We consider $U(\phi) > 0$ only, so $A(\phi) > 0$ at any ϕ.

Now we differentiate (7) over time, substitute (8) and get

$$\begin{aligned} \dot{\phi} &= \psi\,, \\ \dot{\psi} &= -3P\sqrt{U}\psi - \frac{A'}{2A}\psi^2 - \frac{V'_{eff}}{A}. \end{aligned} \tag{9}$$

De Sitter solutions corresponds to $\psi = 0$, hence, $V'_{eff}(\phi_{dS}) = 0$, in other words

$$V'(\phi_{dS})U(\phi_{dS}) = 2V(\phi_{dS})U'(\phi_{dS}).$$

The corresponding Hubble parameter is

$$H_{dS} = P_{dS}\sqrt{U(\phi_{dS})} = \pm\sqrt{\frac{V(\phi_{dS})}{6U(\phi_{dS})}} = \pm\sqrt{\frac{V'(\phi_{dS})}{12U'(\phi_{dS})}}\,. \tag{10}$$

Lyapunov's stability

Let us consider Lyapunov's stability of a de Sitter solution. Substituting

$$\phi(t) = \phi_{dS} + \phi_1(t), \qquad \psi(t) = \psi_1(t), \tag{11}$$

into (9), we get the following linear system on $\phi_1(t)$ and $\psi_1(t)$:

$$\begin{aligned} \dot{\phi}_1 &= \psi_1, \\ \dot{\psi}_1 &= -3H_{dS}\psi_1 - \frac{V''_{eff}(\phi_{dS})}{A(\phi_{dS})}\phi_1. \end{aligned} \tag{12}$$

So, the considering de Sitter solution is stable under conditions $H_{dS} > 0$ and $V''_{eff}(\phi_{dS}) > 0$. In other words, the model has a stable de Sitter solution only if the potential V_{eff} have a minimum. Note that this conclusion are valid for an arbitrary differentiable functions U and V, under condition $U(\phi_{dS}) > 0$.

II. MODELS WITH $V(\phi) < 0$.

The simplest way to get a non-positive definite potential from the known positive definite one is to subtract a positive constant.
Let us consider such a potential that $\exists\phi$: $V(\phi) < 0$.
Equation (2) has the following solutions:

$$H_{\pm} = -\frac{\dot{U}}{2U} \pm \sqrt{\frac{1}{6U}\left[\frac{\dot{\phi}^2}{2} + \frac{3U'^2\dot{\phi}^2}{2U} + V\right]}.$$

The function H is a continuous function, so, if $\forall\phi$: $V(\phi) > 0$, then evolution of the Universe in such a model is described either only H_{-} or only H_{+}.
It depends on initial conditions.

On the $(\phi, \dot{\phi})$ plane there is the boundary, at any point of which

$$\frac{\dot{\phi}^2}{2} + \frac{3(U'\,\dot{\phi})^2}{2U} + V = 0\,. \tag{13}$$

This boundary divides the phase plane into two domains: one corresponds to real values of the Hubble parameter $H_{\pm}$, the other one corresponds to non-real values of this function. We call the domain on the $(\phi, \dot{\phi})$ plane, which corresponds to non-real values of the Hubble parameter as "unreachable domain".

The boundary of this domain is defined by (13).

If $V(\phi)$ is not a positive definite function, then it is possible that a part of evolution is described by H_+, whereas the other part — by H_-.

A new variable Q

Let us introduce

$$Q \equiv H + \frac{\dot{U}}{2U} = H + \frac{U'\dot{\phi}}{2U}. \quad (14)$$

The boundary is the line $Q = 0$.

If a trajectory starts from a real value of H, then it never crosses the line $Q = 0$, but can touch this line.

$Q(t)$ is a monotonically decreasing function.

If $H(t_0) = H_-$, then $Q < 0$ always.

If $H(t_0) = H_+$, so, $Q(t_0) > 0$, then $\exists t_1$ such that $Q(t_1) = 0$.

The evolution of the Universe in such a model is described by H_+ at $t < t_1$ and H_- at $t > t_1$.

We consider

$$V(\phi) = \frac{\varepsilon}{4}\left(\phi^2 - b^2\right)^2 - \Lambda,$$

where $\varepsilon > 0$, b and $\Lambda > 0$ are constants.

We choose such values of these constant that $V(0) > 0$.

I.Ya. Aref'eva, N.V. Bulatov, R.V. Gorbachev, S.Yu. V., *Class. Quant. Grav.* **31** (2014) 065007, arXiv:1206.2801

In terms of the variable Q, equations (2) and (6) take the following form

$$3Q^2 = \frac{\dot\phi^2}{4U} + \frac{3\dot U^2}{4U^2} + \frac{V}{2U}. \tag{15}$$

$$\dot Q - \frac{\dot U}{2U}Q = -\frac{U + 3U'^2}{4U^2}\,\dot\phi^2. \tag{16}$$

Therefore,

$$\frac{d}{dt}\left[\frac{Q}{\sqrt{U}}\right] = -\frac{U + 3U'^2}{4U^2\sqrt{U}}\,\dot\phi^2 \leqslant 0. \tag{17}$$

For any $U(\phi) > 0$ the function $Q/\sqrt{U}$ decrease monotonically.
If for some moments of time t_1 and $t_2 > t_1$ we have $\phi(t_2) = \phi(t_1)$ and $\phi(t)$ is not a constant at $t_1 \leqslant t \leqslant t_2$, then $Q(t_2) < Q(t_1)$.
The physical reasons of inequality (17) will be clear when we consider this model in the Einstein frame.

INDUCED GRAVITY MODEL

Let

$$U(\phi) = \frac{1}{2}\xi\phi^2\,, \qquad \xi > 0.$$

Equation (2) can be rewritten as follows:

$$H^2 + 2H\frac{\psi}{\phi} - \frac{V}{3\xi\phi^2} - \frac{1}{6\xi}\left(\frac{\psi}{\phi}\right)^2 = 0 \tag{18}$$

and has the solutions

$$H_{\pm} = -\frac{\psi}{\phi} \pm \sqrt{\left(1 + \frac{1}{6\xi}\right)\left(\frac{\psi}{\phi}\right)^2 + \frac{V}{3\xi\phi^2}}. \tag{19}$$

The function H is a continuous function, so, if $V(\phi) > 0$ for all ϕ, then evolution of the Universe in such a model is described either only H_- or only H_+. It depends on initial conditions.
If $V(\phi)$ is not a positive definite function, then it is possible that a part of evolution is described by H_+, whereas the other part — by H_-.

HIGGS-LIKE POTENTIAL

We have the system of three first order differential equations

$$\dot{\phi} = \psi\,, \tag{20}$$

$$\dot{\psi} = \; -3H\psi - \frac{\psi^2}{\phi} + \frac{1}{(1+6\xi)\phi}[4V(\phi) - \phi V'(\phi)], \tag{21}$$

$$\dot{H} = \frac{4H\psi}{(1+6\xi)\phi} + \frac{V'(\phi)}{(1+6\xi)\phi} - \frac{12\xi}{1+6\xi}H^2 - \frac{1+2\xi}{2\xi(1+6\xi)}\left(\frac{\psi}{\phi}\right)^2. \tag{22}$$

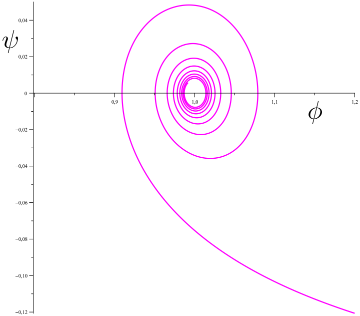
Equation (18) is a condition of the initial data of system (20)–(22).
Let us subtract the cosmological constant from the Higgs-like potentials:

$$V(\phi) = \frac{\varepsilon}{4}\left(\phi^2 - b^2\right)^2 - \Lambda, \tag{23}$$

where $\varepsilon > 0$, b and $\Lambda > 0$ are constants.

NUMERIC SOLUTIONS AT $\Lambda = 0$

At $\Lambda = 0$ the behaviour of solutions is well-known:



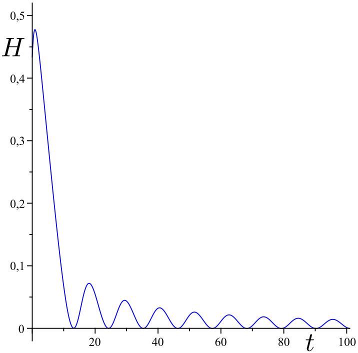


Figure: The solution of system (20)–(22) at $\Lambda = 0$. We choose $b = 1$, $\varepsilon = 10$, $\xi = 10$. The initial conditions are $\phi_0 = 2$, $\psi_0 = 0$, H_0 is calculated by (19) with sing "$+$" ($H_0 = \sqrt{3}/4$). The Hubble parameter is always H_+.

In I.Ya. Aref'eva, N.V. Bulatov, R.V. Gorbachev, S.Yu. V., *Class. Quant. Grav.* **31** (2014) 065007, we get numeric solutions at $\Lambda > 0$.

NUMERIC SOLUTIONS AT $\Lambda > 0$

Numeric calculations give the following solution for system (20)–(22):

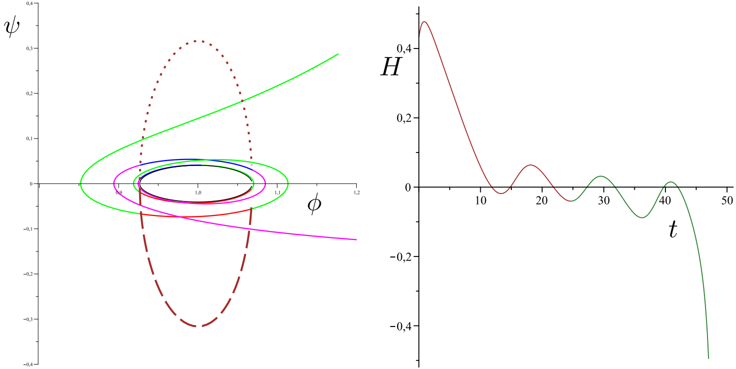


Figure: We choose $\Lambda = 0.05$, $b = 1$, $\epsilon = 10$, $\xi = 10$.
The initial conditions are $\phi_0 = 2$, $\psi_0 = 0$, $H_0 = H_{0+}$.
On the left picture, brown dashed line corresponds to $H_+ = 0$, brown dashed line with long dashes corresponds to $H_- = 0$, black line is the boundary of the unreachable domain. On the right picture, brown color means that $H = H_+$, whereas $H = H_-$ is drown in dark green color.

For the induced gravity $Q \equiv H + \dfrac{\psi}{\phi}$.

We consider $\phi > 0$.

$$\frac{d}{dt}\left[\frac{Q}{\phi}\right] = -\frac{6\xi + 1}{2\xi\phi}\left(\frac{\psi}{\phi}\right)^2 \leqslant 0. \tag{24}$$

Let at t_1 and $t_2 > t_1$ we have $\phi(t_2) = \phi(t_1)$, from (24) we get:

$$\frac{Q(t_2)}{\phi(t_2)} - \frac{Q(t_1)}{\phi(t_1)} = \frac{1}{\phi(t_1)}(Q(t_2) - Q(t_1)) = -\frac{6\xi + 1}{2\xi}\int\limits_{t_1}^{t_2} \frac{\psi^2}{\phi^3}\,dt \leqslant C_0 < 0.$$

So, for any circle value of Q decreases on some positive value, which doesn't tend to zero, when number of circles tends to infinity, hence, only a finite number of circles is necessary to get the value $Q = 0$.

For potential (23) equation $Q = 0$ is equivalent to

$$(1+6\xi)\dot{\phi}^2 = 2\Lambda - \frac{\varepsilon}{2}\left(\phi^2 - b^2\right)^2.$$

At $\Lambda < \varepsilon b^4/4$ the unreachable domain consists of two separated parts.
This curve is not a solution.
To prove it we consider the equation without potential

$$\dot{H} = -\frac{\dot{\psi}}{\phi} + H\frac{\psi}{\phi} - \frac{2\xi+1}{2\xi}\left(\frac{\psi}{\phi}\right)^2.$$

Substituting $H = -\dfrac{\psi}{\phi}$, we get

$$\frac{6\xi+1}{\xi}\left(\frac{\psi}{\phi}\right)^2 = 0.$$

So, only a constant solution with $\psi = 0$ and $H = 0$ can belong to the boundary.

We proved the following statements:

- **If a solution tends to the boundary, then it reaches the boundary in a finite time.**
- **The boundary is not a solution.**
- **On the boundary $\dot{Q} < 0$, $H(t)$ evolves from H_+ to H_-.**
- **The phase trajectories are being attracted to the boundary of the unreachable domain, touch it and go to infinity:**

Solutions with $H(t) = H_+$ and $H(t) = H_-$.

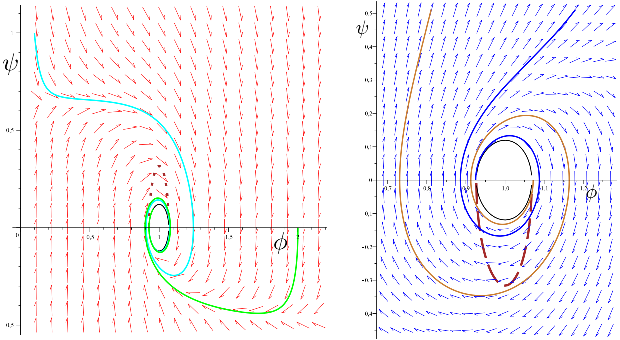
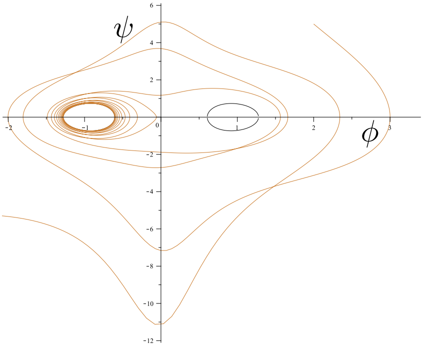
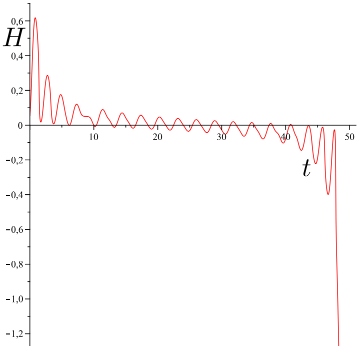


Figure: Phase portraits of system (20)–(21) at $\Lambda = 0.05$, $b = 1$, $\epsilon = 10$, $\xi = 1$. The function $H(t)$ is given by (18) as H_+ (left) and H_- (right).

Let us consider the model with

$$U(\phi) = \frac{\xi}{2}\phi^2 + \frac{M_{\rm Pl}^2}{16\pi}.$$





If a trajectory rotates around one of the unreachable domains starting from H_+ then it will surely reach the boundary of the unreachable domain for a finite period of time and the evolution corresponding to H_+ changes to the evolution with H_-.

EXACT SOLUTIONS

In $R + R^2$ gravity there exits the well-known Ruzmaikin solution with $H = C(t - t_0)$. T.V. Ruzmaikina, A.A. Ruzmaikin, *JETP* **30** (1970) 372
At $C < 0$ this solution is used for the Starobinsky inflationary model:
A.A. Starobinsky, *JETP Lett.* **30** (1979) 682 [*Pisma Zh. Eksp. Teor. Fiz.* **30** (1979) 719].
There is the connection due to the conformal transformation:

$f(R) \quad \leftrightarrow \quad$ GR + a scalar field $\quad \leftrightarrow \quad$ a nonminimal coupled scalar field.

It is easy to see that for $U(\phi) = 1/6 - \xi\phi^2$ Eq. (6) has the solution

$$H = H_0 t + C_0, \qquad \phi = \frac{H_0 t + C_0}{\sqrt{3H_0(2\xi - 1)}}. \tag{25}$$

The corresponding potential is

$$V(\phi) = H_0 \left(9(1 - 2\xi)\xi\phi^4 - \frac{3(1 + 2\xi)}{2}\phi^2 - \frac{1}{6(2\xi - 1)}\right). \tag{26}$$

To get real $\phi(t)$ at $\xi < 0$ we should put $H_0 < 0$. If $C_0 > 0$, then the inflationary scenario can be realize. At the first moment, $t = 0$, $H = C_0$ and then tends to zero. Note that $V(0) < 0$.
M.A. Skugoreva, A.V. Toporensky, S.Yu. Vernov, arXiv:1404.6226

These two frames are related by conformal transformation $g_{\mu\nu} = \Omega^2 g^{(E)}_{\mu\nu}$:

$$\Rightarrow \quad R = \Omega^{-2}\left[R^{(E)} - 6\left(\Box^{(E)}\ln\Omega + g^{\mu\nu(E)}\nabla^{(E)}_{\mu}\ln\Omega\nabla^{(E)}_{\nu}\ln\Omega\right)\right]$$

$$\text{At} \qquad \Omega^{-2} = \frac{\kappa^2}{2}U \qquad \rightarrow \qquad \Omega = \frac{\sqrt{2}}{\kappa\sqrt{U}},$$

where $\kappa^2 \equiv 8\pi/{M_{\rm Pl}}^2$. We get the model with a minimally coupled scalar field and the corresponding FLRW metric has the interval

$$ds^2 = \ -dt_E^2 + a_E^2(t_E)\delta_{ij}dx^i dx^j\ , \tag{27}$$

$$dt_E = \Omega^{-1}dt = \frac{\kappa\sqrt{U}}{\sqrt{2}}dt, \qquad a_E = \frac{\kappa\sqrt{U}}{\sqrt{2}}a\,,$$

$$H_E \equiv \frac{d\log a_E}{dt_E} = \Omega\left(H - \frac{\dot{\Omega}}{\Omega}\right) = \frac{\sqrt{2}}{\kappa\sqrt{U}}\left(H + \frac{\dot{U}}{2U}\right) = \frac{\sqrt{2}}{\kappa\sqrt{U}}Q.$$

$Q = 0$ is equivalent to $H_E = 0$.

The use of the FLRW metric essentially simplify the Einstein equations. But, only a few cosmological models with scalar fields are integrable.
P. Fré, A. Sagnotti, A.S. Sorin, *Nucl. Phys.* B **877** (2013) 1028, arXiv:1307.1910.
The standard way to integrate a cosmological model is

- to use the FLRW metric with a parametric time

$$ds^2 = N^2(\tau)d\tau^2 - a^2(\tau)\left(dx_1^2 + dx_2^2 + dx_3^2\right).$$

- to guess a suitable lapse function $N(\tau)$.
- to linearize equations, introducing new depending variables.

Our goal is to find integrable model with non-minimal coupling using the knowledge of integrable models with minimal coupling.
To do this we use the FLRW metric with a parametric time and find the correspondence between potentials and lapse functions in the Einstein and Jordan frames.

$$\frac{6U\dot{a}^2}{a^2}+\frac{6U'\dot{a}\dot{\phi}}{a}=\frac{1}{2}\dot{\phi}^2+N^2V. \tag{28}$$

$$\frac{4U\ddot{a}}{a}+\frac{2U\dot{a}^2}{a^2}+\frac{4U'\dot{a}\dot{\phi}}{a}-\frac{4U\dot{a}\dot{N}}{aN}+2U''\dot{\phi}^2+2U'\ddot{\phi}-\frac{2U'\dot{\phi}\dot{N}}{N}=-\frac{1}{2}\dot{\phi}^2+N^2V. \tag{29}$$

The variation with respect to ϕ gives the Klein–Gordon equation:

$$\ddot{\phi}+\left(3\frac{\dot{a}}{a}-\frac{\dot{N}}{N}\right)\dot{\phi}-6U'\left[\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right]+6\frac{\dot{a}\dot{N}}{aN}U'+N^2V'=0. \tag{30}$$

A suitable combination of these equations is

$$\left[\ddot{\phi}+\left(3\frac{\dot{a}}{a}-\frac{\dot{N}}{N}\right)\dot{\phi}\right]\left[1+3\frac{U'^2}{U}\right]+\frac{U'}{2U}\dot{\phi}^2\left[1+6U''\right]+N^2\left[V'-2\frac{U'}{U}V\right]=0.$$

CONFORMAL TRANSFORMATION

Let us make the conformal transformation of the metric

$$g_{\mu\nu} = \frac{U_0}{U}\tilde{g}_{\mu\nu},$$

where U_0 is a constant, and introduce a new scalar field ϕ such that

$$\frac{d\tilde{\varphi}}{d\phi} = \frac{\sqrt{U_0(U + 3U'^2)}}{U} \quad \Rightarrow \quad \tilde{\varphi} = \int \frac{\sqrt{U_0(U + 3U'^2)}}{U} d\phi. \tag{31}$$

In this case the action (1) becomes the action for a minimally coupled scalar field:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[U_0 R(\tilde{g}) - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\varphi}_{,\mu}\phi_{,\nu} + W(\tilde{\varphi})\right], \tag{32}$$

where

$$W(\tilde{\varphi}) = \frac{U_0^2 V(\phi(\tilde{\varphi}))}{U^2(\phi(\tilde{\varphi}))}. \tag{33}$$

The Friedmann metric becomes $ds^2 = \tilde{N}^2 d\tau^2 - \tilde{a}^2 \vec{dl}^2$, where

$$\tilde{N} = \sqrt{\frac{U}{U_0}}N, \qquad \tilde{a} = \sqrt{\frac{U}{U_0}}a.$$

We have the following equations:

$$6U_0\tilde{h}^2 = \frac{1}{2}\dot{\tilde{\varphi}}^2 + \tilde{N}^2 W, \tag{34}$$

$$4U_0\dot{\tilde{h}} + 6U_0\tilde{h}^2 - 4U_0\tilde{h}\frac{\dot{\tilde{N}}}{\tilde{N}} = -\frac{1}{2}\dot{\tilde{\varphi}}^2 + \tilde{N}^2 W, \tag{35}$$

$$\ddot{\tilde{\varphi}} + \left(3\tilde{h} - \frac{\dot{\tilde{N}}}{\tilde{N}}\right)\dot{\phi} + \tilde{N}^2 W_{,\phi} = 0, \tag{36}$$

where $\tilde{h} \equiv \dot{\tilde{a}}/\tilde{a}$.

Let us suppose that for some potential W we know the general exact solution of the system of equations (34)–(36): $\tilde{\varphi}(\tau)$, $\tilde{a}(\tau)$, $\tilde{N}(\tau)$.
We also suppose that the function $\phi(\tilde{\varphi})$ is known explicitly.
In this case, we can also find the general solution of the system of equations (28)–(30) with the potential

$$V(\phi) = \frac{U^2(\phi)W(\tilde{\varphi}(\phi))}{U_0^2}\,, \tag{37}$$

To do it we really need only

$$N(\tau) = \sqrt{\frac{U_0}{U(\phi(\tilde{\varphi}(\tau))}}\tilde{N}(\tau).$$

It is the most important information.
After this we consider only equations in the Jordan frame and linearize them.

Let us consider the induced gravity with

$$U(\phi) = \frac{1}{2}\xi\phi^2. \qquad (38)$$

In this model

$$\tilde{\varphi} = \sqrt{\frac{2U_0(1+6\xi)}{\xi}}\ln\left[\frac{\phi}{\phi_0}\right] \quad \text{and} \quad \phi = \phi_0 e^{\sqrt{\frac{\xi}{2U_0(1+6\xi)}}\tilde{\varphi}}. \qquad (39)$$

We put $\xi \neq -1/6$, because at $\xi = -1/6$ we have $U + 3U'^2 = 0$ and nontrivial solutions exist for the potential $V = V_0\phi^4$ only.
(I.Ya. Aref'eva, N.V. Bulatov, R.V. Gorbachev, S.Yu.V., arXiv:1206.2801)
In the case $\xi = -1/6$ we consider models with

$$U(\phi) = U_0 - \frac{\phi^2}{12}, \qquad (40)$$

(K. Bamba, Sh. Nojiri, S.D. Odintsov, D. Sáez-Gómez, arXiv:1401.1328)
In this case

$$\tilde{\varphi} = \sqrt{3U_0}\ln\left[\frac{\sqrt{12U_0}+\phi}{\sqrt{12U_0}-\phi}\right] \quad \text{and} \quad \phi = \sqrt{12U_0}\tanh\left[\frac{\tilde{\varphi}}{\sqrt{12U_0}}\right]. \qquad (41)$$

Let us consider the cosmological model with a minimally coupled scalar field and the exponential potential:

$$W = W_0 e^{2\sqrt{3}\lambda\tilde{\varphi}}, \tag{42}$$

where $\lambda \neq \pm 1$.
D.S. Salopek and J.R. Bond, *Phys. Rev.* D **42** (1990) 3936–3962.
We put $U_0 = 1/4$.
In the induced gravity model the corresponding potential is

$$V(\phi) = 4W_0\xi^2\phi^4\left(\frac{\phi}{\phi_0}\right)^{\lambda\sqrt{\frac{6(1+6\xi)}{\xi}}} = 4W_0\xi^2\phi^4\left(\frac{\phi}{\phi_0}\right)^{6\lambda\Gamma}.$$

where $\Gamma \equiv \sqrt{\frac{1+6\xi}{6\xi}}$.
In the model including the Hilbert–Einstein curvature term plus a scalar field conformally coupled to gravity

$$\mathcal{V} = W_0\left[1-\frac{\phi^2}{3}\right]^2\left(\frac{\sqrt{3}+\phi}{\sqrt{3}-\phi}\right)^{3\lambda} = W_0\Theta\Upsilon^{3\lambda}, \quad \Theta \equiv \left[1-\frac{\phi^2}{3}\right]^2, \Upsilon \equiv \frac{\sqrt{3}+\phi}{\sqrt{3}-\phi}.$$

Table: FAMILIES POTENTIALS OF INTEGRABLE MODELS

W (minimal coupling)	V (induced gravity)	$\mathcal{V}$ (conformal coupling)
$c_0 e^{2\sqrt{3}\lambda\tilde{\varphi}}$	$\tilde{c}_0 \phi^{4+6\lambda\Gamma}$	$c_0 \Theta \Upsilon^{3\lambda}$
$c_0 + c_1 e^{\sqrt{3}\tilde{\varphi}} + c_2 e^{-\sqrt{3}\tilde{\varphi}}$	$\tilde{c}_0\phi^4 + \tilde{c}_1\phi^{4+3\Gamma} + \tilde{c}_2\phi^{4-3\Gamma}$	$\Theta\left[c_0 + c_1\Upsilon^{\frac{3}{2}} + c_2\Upsilon^{-\frac{3}{2}}\right]$
$c_1 e^{2\sqrt{3}\lambda\tilde{\varphi}} + c_2 e^{\sqrt{3}(\lambda+1)\tilde{\varphi}}$	$\tilde{c}_1\phi^{4+6\lambda\Gamma} + \tilde{c}_2\phi^{4+3(\lambda+1)\Gamma}$	$\Theta\left[c_1\Upsilon^{3\lambda} + c_2\Upsilon^{\frac{3}{2}(\lambda+1)}\right]$
$c_1 e^{2\sqrt{3}\tilde{\varphi}} + c_2$	$\phi^4\left[\tilde{c}_1\phi^{6\Gamma} + \tilde{c}_2\right]$	$\Theta\left[c_1\Upsilon^3 + c_2\right]$
$c_0\tilde{\varphi}e^{2\sqrt{3}\tilde{\varphi}}$	$\sqrt{3}\Gamma\tilde{c}_0\phi^{4+6\Gamma}\ln\left[\frac{\phi}{\phi_0}\right]$	$\frac{\sqrt{3}}{2}c_0\Theta\Upsilon^3\ln(\Upsilon)$
$c_1 e^{2\sqrt{3}\lambda\tilde{\varphi}} + c_2 e^{\frac{2\sqrt{3}}{\lambda}\tilde{\varphi}}$	$\phi^4\left[\tilde{c}_1\phi^{6\lambda\Gamma} + \tilde{c}_2\phi^{6\frac{\Gamma}{\lambda}}\right]$	$\Theta\left[c_1\Upsilon^{3\lambda} + c_2\Upsilon^{\frac{3}{\lambda}}\right]$

In Table 1 we present the list of the potentials of integrable cosmological models. The constants $\tilde{c}_i = 4\xi^2 c_i$, $\lambda \neq \pm 1$, $\lambda \neq 0$.

P. Fré, A. Sagnotti, A.S. Sorin, arXiv:1307.1910 (minimal coupling).

Table: Lapse functions for integrable cases

	$\tilde{N}$ (minimal coupling)	N (induced gravity)	$\mathcal{N}$ (conformal coupling)
1	$\frac{\sqrt{6}}{\sqrt{c_0}} e^{-\sqrt{3}\lambda\tilde{\varphi}}$	$\frac{\sqrt{3}}{\sqrt{\xi c_0}} \phi^{-3\lambda\Gamma-1}$	$\sqrt{\frac{18}{c_0(3-\phi^2)}}\Upsilon^{-3\lambda/2}$
2	1	$\frac{\sqrt{2}}{\sqrt{\xi}\phi}$	$\sqrt{\frac{3}{3-\phi^2}}$
3	$e^{-\sqrt{3}\lambda\tilde{\varphi}}$	$\frac{1}{\sqrt{2\xi}}\phi^{-3\Gamma\lambda-1}$	$\sqrt{\frac{3}{3-\phi^2}}\Upsilon^{-3\lambda/2}$
4	$e^{-\sqrt{3}\tilde{\varphi}}$	$\frac{1}{\sqrt{2\xi}}\phi^{-3\Gamma-1}$	$\sqrt{\frac{3}{3-\phi^2}}\Upsilon^{-3/2}$
5	$\frac{e^{-2\sqrt{3}\tilde{\varphi}}}{\tilde{a}^3}$	$\frac{9(\Gamma^2-1)^2}{a^3\phi^4}\left(\frac{\phi}{\phi_0}\right)^{-6\Gamma}$	$\frac{9}{a^3}\frac{(\sqrt{3}-\phi)}{(\sqrt{3}+\phi)^5}$
6	$\tilde{a}^3$	$\frac{\phi^2 a^3}{3(\Gamma^2-1)}$	$\left(1-\frac{\phi^2}{3}\right)^2 a^3$

A.Yu. Kamenshchik, E.O. Pozdeeva, A. Tronconi, G. Venturi, and S.Yu. V., *Class. Quant. Grav.* **31 (2014) 105003**, arXiv:1312.3540

The first Friedmann equation with $U(\phi) = \frac{\xi}{2}\phi^2$ is

$$\left(\frac{d}{d\tau}\ln(a\phi)\right)^2 - \left(\frac{d}{d\tau}\ln(\phi^\Gamma)\right)^2 = \frac{VN^2}{3\xi\phi^2}, \tag{43}$$

where $\Gamma \equiv \sqrt{\frac{1+6\xi}{6\xi}}$.
Let us consider

$$V = 4\xi^2 c_0 \phi^{2n}, \qquad n = 2 + 3\lambda\Gamma. \tag{44}$$

Suitable choice is

$$N = \frac{\sqrt{3}}{\sqrt{\xi c_0}}\phi^{1-n}. \tag{45}$$

We introduce new variables u and v:

$$a\phi \equiv e^{u+v}\ , \qquad \phi^\Gamma \equiv e^{u-v},$$

and obtain Eq. (43) as follows:

$$\dot{u}\dot{v} = \frac{VN^2}{12\xi\phi^2} = 1 \quad \Rightarrow \quad \dot{u} = \frac{1}{\dot{v}}. \tag{46}$$

Let us consider

$$\left[\ddot{\phi} + \left(3\frac{\dot{a}}{a} - \frac{\dot{N}}{N}\right)\dot{\phi} + \frac{\dot{\phi}^2}{\phi}\right](1+6\xi) + \left[V' - \frac{4}{\phi}V\right]N^2 = 0. \quad \Leftrightarrow$$

$$\Gamma\left(\ddot{u} - \ddot{v}\right) + (n-2)(\dot{u} - \dot{v})^2 + 3\Gamma\left(\dot{u}^2 - \dot{v}^2\right) + 4(n-2) = 0. \qquad (47)$$

$$x = \dot{u} \qquad \Rightarrow \qquad \dot{v} = \frac{1}{x}\,,$$

equation (47) is the Riccati equation

$$\dot{x} + \frac{n-2+3\Gamma}{\Gamma}x^2 + \frac{n-2-3\Gamma}{\Gamma} = 0.$$

The standard substitution

$$x = \frac{\Gamma\dot{y}}{(n-2+3\Gamma)y}\,,$$

gives the following linear equation:

$$\ddot{y} + \left(\frac{(n-2)^2}{\Gamma^2} - 9\right)y = 0\,.$$

So, we are able to get the general solution.

The induced gravity cosmological model with power-law potential is integrable.

CONCLUSIONS

- Cosmological models with non-minimally coupling scalar fields has been considered.
- We study dynamics of non-minimally coupled scalar field cosmological models with Higgs-like potentials and a negative cosmological constant.
- In these models the inflationary stage of the Universe evolution changes into a quasi-cyclic stage of the Universe evolution with oscillation behaviour of the Hubble parameter from positive to negative values. The Hubble parameter can perform a few cycles before to become negative forever.
- The exact solution and the corresponding potential are presented.
- We show how to get integrable models with non-minimal coupling using the suitable parametric time.
- We obtain the general solution for one of the integrable models, namely, the induced gravity model with a power-law potential for the self-interaction of the scalar field.
- We show that the knowledge of the suitable lapse function maybe enough to get the general solutions, solving equations in the Jordan frame only.