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Conformal invariance for scalar and Dirac particles in Riemannian spacetimes: new results

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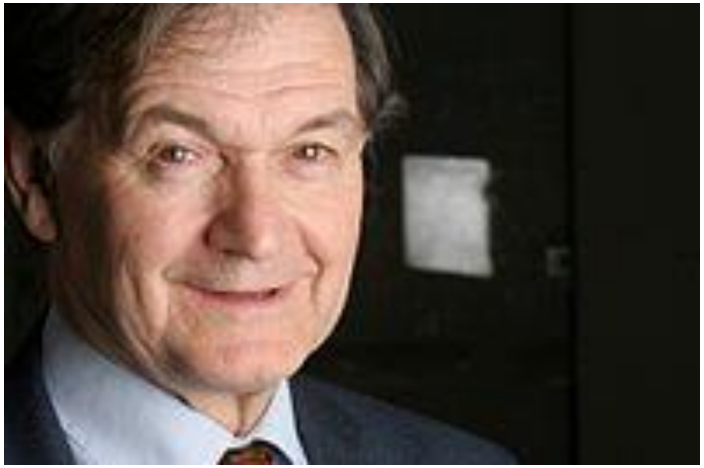
OUTLINE

- **Covariant Klein-Gordon equation and conformal invariance for massless particles. *Fifty-year history***
- **Conformal invariance and new (Hermitian) form of the Klein-Gordon equation. Conformal symmetry for a pointlike scalar particle (Higgs boson)**
- **Conformal symmetries of Hamiltonians**
- **Exact Foldy-Wouthyusen transformation**
- **Inclusion of electromagnetic interactions**
- **Comparison of scalar and Dirac particles**
- **Summary**

Covariant Klein-Gordon equation and conformal invariance for massless particles. *Fifty-year history*

Conformal invariance for a massless particle



R. Penrose

N.A. Chernikov (1928 – 2007)

E. Tagirov

R. Penrose, In: Relativity, Groups and Topology. London: Gordon and Breach, ***1964***, p. 565.

N. Chernikov and E. Tagirov, Ann. Inst. Henri Poincarè **9**, 109 (1968).

Covariant Klein-Gordon equation with the nonminimal coupling:

$$\left(\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu + m^2 - \lambda R \right) \psi = 0, \quad \lambda = \frac{1}{6}.$$

Non-minimal coupling with the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\lambda}{}_{\mu\lambda\nu}.$$

The sign of the Penrose-Chernikov-Tagirov term depends on the definition of R

In: Relativity, Groups and Topology.
London: Gordon and Breach, 1964

Republication of:
Conformal treatment of infinity

Roger Penrose

The utility of this idea rests on the fact that the zero rest-mass free-field equations for each spin value are conformally invariant if interpreted suitably. For example, for spin zero, if the wave equation is written as

$$\left\{\nabla_\mu \nabla^\mu + \frac{R}{6}\right\}\phi = 0$$

where R is the scalar curvature and ∇_μ denotes covariant derivative—both according to the metric $g_{\mu\nu}$ of $\mathscr{M}$, then

$$\left\{\tilde{\nabla}_\mu \tilde{\nabla}^\mu + \frac{\tilde{R}}{6}\right\}\tilde{\phi} = 0$$

where $\tilde{\nabla}_\mu$, $\tilde{R}$ refer to the metric $\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ of $\tilde{\mathscr{M}}$ and where

$$\tilde{\phi} = \Omega\phi.$$

N. Chernikov and E. Tagirov, Ann. Inst. Henri Poincarè **9**, 109 (1968)

$$\Box - \frac{n-2}{4(n-1)} R = \Omega^{\frac{n+2}{2}} \left(\Box' - \frac{n-2}{4(n-1)} R' \right) \Omega^{\frac{2-n}{2}},$$

$$\psi' = \Omega^{\frac{n-2}{2}} \psi, \qquad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}.$$

n is a number of dimensions

But the only discovered pointlike scalar particle (Higgs boson) is massive!

Chernikov and Tagirov discussed an importance of non-minimal coupling with the scalar curvature for massive particles

New step ahead (Hamiltonian approach):

A. Accioly and H. Blas, Exact Foldy-Wouthuysen transformation for real spin-0 particle in curved space, PHYSICAL REVIEW D **66**, 067501 (2002).

Static metric in isotropic coordinates:

$$ds^2 = V(\mathbf{x})^2\left(dx^0\right)^2 - W(\mathbf{x})^2\left(d\mathbf{x}\right)^2.$$

Feshbach-Villars transformation (useless for massless particles):

$$\psi = \phi + \chi, \quad \frac{i}{m}\frac{\partial\psi}{\partial t} = \phi - \chi.$$

the mass in the deniminator!

$$\mathcal{H}_{FW}=\rho_3\sqrt{m^2V^2+F\mathbf{p}^2F-\frac{1}{4}\nabla F\cdot\nabla F+\frac{V^2}{2W^3}\Delta W+\frac{V}{2W^2}\Delta V-\frac{1}{6}V^2R},$$

$$-\frac{1}{6}V^2R=\frac{1}{6}F\Delta F-\frac{V^2}{2W^3}\Delta W-\frac{V}{2W^2}\Delta V,$$

$$\mathcal{H}_{FW}=\rho_3\sqrt{m^2V^2+F\mathbf{p}^2F-\frac{1}{4}\nabla F\cdot\nabla F+\frac{1}{6}F\Delta F},\quad F=\frac{V}{W}.$$

$$\text{If}\quad m=0,\quad \mathcal{H}_{FW}\left((g')^{\mu\nu}=\frac{g^{\mu\nu}}{\Omega(\mathbf{x})^2}\right)=\mathcal{H}_{FW}\left(g^{\mu\nu}\right).$$

$$ds^2=V^2\left(dx^0\right)^2-W^2\left(d\mathbf{x}\right)^2.$$

All terms except for the first term are conformally invariant

Conformal transformation changes only such terms in the Foldy-Wouthuysen Hamiltonian which are proportional to the particle mass

But it is only a shadow of the conformal invariance, because *m*≠0!

Hamiltonian approach in classical general relativity:

$$\mathcal{H}_{class} = \sqrt{\frac{m^2 - G^{ij} p_i p_j}{g^{00}}} + \frac{g^{0i} p_i}{g^{00}}, \quad G^{ij} = g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}.$$

G. Cognola, L. Vanzo, and S. Zerbini, Gen. Relativ. Gravit. **18**, 971 (1986).

$$\text{If} \quad m = 0, \quad \mathcal{H}_{class}\left((g')^{\mu\nu} = \frac{g^{\mu\nu}}{\Omega(\mathbf{x})^2} \right) = \mathcal{H}_{class}\left(g^{\mu\nu}\right).$$

The second-order form of this classical Hamiltonian equation is

$$g^{\mu\nu} p_\mu p_\nu - m^2 + \cancel{\lambda R} = 0, \qquad \lambda = 0!$$

The classical equations contrary to quantum mechanical ones correspond to the minimal coupling

PHYSICAL REVIEW D **88**, 045004 (2013)

Scalar particle in general inertial and gravitational fields and conformal invariance revisited

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1. **Exact Foldy-Wouthuysen transformation for**
   i) general static metric
   ii) frame rotating in the Kerr field approximated by a spatially isotropic metric
2. **Foldy-Wouthuysen Hamiltonian is conformally invariant for massless particles and conformallly symmetric for massive ones**
3. **Proof of similarity of conformal transformations for scalar and Dirac particles**

Conformal invariance and new (Hermitian) form of the Klein-Gordon equation. Conformal symmetry for a pointlike scalar particle

(Higgs boson)

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu+m^2-\lambda R\right)\psi=0.$$

$$\psi'=\Omega^{\frac{n-2}{2}}\psi,\qquad g'_{\mu\nu}=\Omega^2 g_{\mu\nu}.$$

The nonunitary transformation

$$\Phi=\sqrt{g^{00}\sqrt{-g}}\,\psi,\quad g=\det g_{\mu\nu},\quad g'=\Omega^{2n}g.$$

Φ is *invariant* relative to conformal transformations

We multiply the KFG equation from left by $\sqrt{\frac{\sqrt{-g}}{g^{00}}}$.

Hermitian form of the KFG equation:

$$\left(\frac{1}{\sqrt{g^{00}\sqrt{-g}}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\frac{1}{\sqrt{g^{00}\sqrt{-g}}}-\frac{\lambda R}{g^{00}}+\frac{m^2}{g^{00}}\right)\Phi=0.$$

For a massless particle

$$\left(\frac{1}{\sqrt{g^{00}\sqrt{-g}}}\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu\frac{1}{\sqrt{g^{00}\sqrt{-g}}}-\frac{\lambda R}{g^{00}}\right)'$$

$$=\left(\frac{1}{\sqrt{g^{00}\sqrt{-g}}}\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu\frac{1}{\sqrt{g^{00}\sqrt{-g}}}-\frac{\lambda R}{g^{00}}\right).$$

This part of the KG equation can be transformed:

Denotations:

$$f = \sqrt{g^{00}\sqrt{-g}}, \qquad \Gamma^{i} = \sqrt{-g}\, g^{0i},$$

$$G^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}, \qquad \Upsilon = \frac{1}{2f}\left\{\partial_i, \Gamma^i\right\}\frac{1}{f} = \frac{1}{2}\left\{\partial_i, \frac{g^{0i}}{g^{00}}\right\},$$

$$\Lambda = -\frac{f_{,0,0}}{f} - \left(\frac{g^{0i}}{g^{00}}\right)_{,i}\frac{f_{,0}}{f} - 2\,\frac{g^{0i}}{g^{00}}\frac{f_{,0,i}}{f} - \left(\frac{g^{0i}}{g^{00}}\right)_{,0}\frac{f_{,i}}{f}$$

$$-\frac{1}{2}\left(\frac{g^{0i}}{g^{00}}\right)_{,0,i} - \frac{1}{2f^2}\left(\frac{g^{0i}}{g^{00}}\right)_{,i}\Gamma^{j}_{,j} - \frac{g^{0i}}{2f^2 g^{00}}\Gamma^{j}_{,i,j}$$

$$+\frac{1}{4f^4}\left(\Gamma^{i}_{,i}\right)^2 - \left(\frac{G^{ij}}{g^{00}}\right)_{,i}\frac{f_{,j}}{f} - \frac{G^{ij}}{g^{00}}\frac{f_{,i,j}}{f} - \frac{\lambda R}{g^{00}}.$$

Equivalent Hermitian form of the KG equation:

$$\left[\left(\partial_0 + \Upsilon\right)^2 + \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \Lambda + \frac{m^2}{g^{00}}\right]\Phi = 0.$$

Υ, G^{ij}/g^{00}, and Λ are invariant relative to conformal transformations

We can extend conformal symmetry on massive particles. The conformal–like transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m$$

conserves the operator [...] acting on Φ and, therefore, conserves Φ when λ=1/6 (generally, (n-2)/[4(n-1)]). As a result, the operator acting on Φ in the equation

$$\left(\frac{1}{\sqrt{g^{00}\sqrt{-g}}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \frac{1}{\sqrt{g^{00}\sqrt{-g}}} - \frac{\lambda R}{g^{00}} + \frac{m^2}{g^{00}} \right) \Phi = 0$$

remains unchanged. The initial covariant Klein-Gordon equation

$$\left(\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu + m^2 - \frac{n-2}{4(n-1)} R \right) \psi = 0$$

has following properties:

$$\left(\Box + m^2 - \frac{n-2}{4(n-1)} R\right)$$

$$= \Omega^{\frac{n+2}{2}} \left(\Box' + m^2 - \frac{n-2}{4(n-1)} R' \right) \Omega^{\frac{2-n}{2}},$$

$$\psi' = \Omega^{\frac{n-2}{2}} \psi \quad \text{when} \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m.$$

The conformal symmetry has extended on a pointlike scalar particle (Higgs boson)!

Conformal symmetries of Hamiltonians

Feshbach-Villars transformation to a Hamiltonian form for a *massive* particle

$$\left[\left(\partial_0 + \Upsilon\right)^2 + T\right]\Phi = 0, \qquad T = \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \Lambda + \frac{m^2}{g^{00}}$$

$$\Phi = \phi + \chi, \quad i\left(\partial_0 + \Upsilon\right)\Phi = m(\phi - \chi).$$

***m* appears in a denominator!**

Generalized Feshbach-Villars transformation for both massive and massless particles

The wave function in the Feshbach-Villars representation is given by

$$\Psi = \frac{1}{2}\begin{pmatrix} \Phi + \frac{i}{m}\left(\partial_0 + \Upsilon\right)\Phi \\ \Phi - \frac{i}{m}\left(\partial_0 + \Upsilon\right)\Phi \end{pmatrix}.$$

It was proved that a similar transformation can be performed with the use of any nonzero parameter instead of the particle mass, *m*

A. Mostafazadeh, *J. Phys. A*, **31**, 7829–7845 (1998); *Ann. Phys.*, **309**, 1–48 (2004).

Successive generalized Feshbach-Villars and Foldy-Wouthyusen transformations

The method has been developed in

A.J. Silenko, Hamilton operator and the semiclassical limit for scalar particles in an electromagnetic field,
Theor. Math. Phys. **156,** 1308 (2008).

$$\psi = \phi + \chi, \quad \frac{i}{N}\left(\partial_0 + \Upsilon\right)\psi = \phi - \chi.$$

N **is an arbitrary nonzero real parameter**

$$i\left(\partial_0 + \Upsilon\right)\Psi = \left[\frac{N}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} + \frac{T}{2N}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\right]\Psi, \quad \Psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}.$$

The Pauli matrices ρ_i **can be used**

$$i\left(\partial_0 + \Upsilon\right)\Psi = \frac{1}{2N}\left[\rho_3\left(N^2 + T\right) + i\rho_2\left(-N^2 + T\right)\right]\Psi.$$

Therefore, we obtain the following generalized Feshbach-Villars Hamiltonian:

$$\mathcal{H}_{gFV}=\rho_3\frac{N^2+T}{2N}+i\rho_2\frac{-N^2+T}{2N}-i\Upsilon.$$

This Hamiltonian *is not changed* by the conformal–like transformation

$$g'_{\mu\nu}=\Omega^2 g_{\mu\nu},\quad m'=\Omega^{-1}m.$$

General method of the Foldy-Wouthyusen transformation

$$\mathcal{H}_{gFV}=\rho_3\mathcal{M}+\mathcal{E}+\mathcal{O},\quad \rho_3\mathcal{M}=\mathcal{M}\rho_3,\quad \rho_3\mathcal{E}=\mathcal{E}\rho_3,\quad \rho_3\mathcal{O}=-\mathcal{O}\rho_3,$$

$$\mathcal{M}=\frac{N^2+T}{2N},\quad \mathcal{E}=-i\Upsilon,\quad \mathcal{O}=i\rho_2\frac{-N^2+T}{2N},$$

$$U=\frac{\varepsilon+\mathcal{M}+\rho_3\mathcal{O}}{\sqrt{2\varepsilon\left(\varepsilon+\mathcal{M}\right)}},\quad \varepsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2},\quad \mathcal{F}=\mathcal{E}-i\frac{\partial}{\partial t}.$$

In the case under consideration,

$$U=\frac{\varepsilon+N+\rho_1\left(\varepsilon-N\right)}{2\sqrt{\varepsilon N}},\quad \varepsilon=\sqrt{T},\quad \mathcal{H}'=\rho_3\varepsilon+\mathcal{E}'+\mathcal{O}',$$

$$\mathcal{E}'=\mathcal{E}+\frac{1}{2\sqrt{\varepsilon}}\left[\sqrt{\varepsilon},\left[\sqrt{\varepsilon},\mathcal{F}\right]\right]\frac{1}{\sqrt{\varepsilon}},\quad \mathcal{O}'=\rho_1\frac{1}{2\sqrt{\varepsilon}}\left[\varepsilon,\mathcal{F}\right]\frac{1}{\sqrt{\varepsilon}}.$$

The transformed (intermediate) Hamiltonian describing the both massive and massless particles does not contain *N* and *is not changed* by the conformal–like transformation!

Final approximate Foldy-Wouthuysen Hamiltonian

$$\mathcal{H}_{FW} = \rho_3 \varepsilon + \mathcal{E}', \qquad \varepsilon = \sqrt{T}.$$

The conformal–like transformation does not change the Foldy-Wouthuysen Hamiltonian!

Conformal symmetries of the generalized Feshbach-Villars and Foldy-Wouthuysen Hamiltonians have been proved
in the general form!

Exact Foldy-Wouthyusen transformation

Sufficient condition of exact Foldy-Wouthuysen transformation

$$[\mathcal{M},\mathcal{O}]=[\mathcal{F},\mathcal{O}]=0 \quad \Rightarrow \quad \mathcal{H}_{FW}=\rho_3\sqrt{T}-i\Upsilon.$$

1. Foldy-Wouthuysen transformation is exact for *any static metric*

In many important cases, the spacetime metric can be represented in a static form with an appropriate coordinate transformation. For example, it can be made for de Sitter and anti-de Sitter spaces)

Static metric in isotropic coordinates

Exact Foldy-Wouthuysen transformation has been performed by Accioly and H. Blas (2002) for massive particles

$$ds^2=V(\mathbf{x})^2\left(dx^0\right)^2-W(\mathbf{x})^2\left(d\mathbf{x}\right)^2.$$

The obtained result formally coincides with that by *Accioly and Blas*

$$\mathcal{H}_{FW}=\rho_3\sqrt{m^2V^2+F\mathbf{p}^2F-\frac{1}{4}\nabla F\cdot\nabla F+\frac{V^2}{2W^3}\Delta W+\frac{V}{2W^2}\Delta V-\frac{1}{6}V^2R},$$

$$-\frac{1}{6}V^2R=\frac{1}{6}F\Delta F-\frac{V^2}{2W^3}\Delta W-\frac{V}{2W^2}\Delta V,$$

$$\mathcal{H}_{FW}=\rho_3\sqrt{m^2V^2+F\mathbf{p}^2F-\frac{1}{4}\nabla F\cdot\nabla F+\frac{1}{6}F\Delta F},\quad F=\frac{V}{W}.$$

But we already have a right to consider the case of *m*=0 showing the conformal invariance!

$$\text{If}\quad m=0,\quad \mathcal{H}_{FW}\left(\left(g'\right)^{\mu\nu}=\frac{g^{\mu\nu}}{\Omega(\mathbf{x})^2}\right)=\mathcal{H}_{FW}\left(g^{\mu\nu}\right).$$

2. Frame rotating in the Kerr field approximated by a spatially isotropic metric

All effects of the Schwarzschild gravitational field and the frame rotation are *exactly* described in *isotropic* Arnowitt-Deser-Misner coordinates

Effects of rotation of the *Kerr* source are described in these coordinates within terms of order of $O(a^2r^{-4})$

total mass M, total angular momentum $J=Mca$

This case covers an observer on the ground of the Earth or on a satellite. It reproduces not only the well-known effects of the rotating frame but also the Lense-Thirring effect.

Frame rotating in the Kerr field: An approximation by a spatially isotropic metric

$$d\mathbf{x}' = d\mathbf{x} - \left[\boldsymbol{\Omega}(r)\times\mathbf{x}\right]dx^0, \quad \boldsymbol{\Omega}(r) = \boldsymbol{\omega}(r) - \mathbf{o}, \quad \mathbf{o} = const,$$

$$ds^2 = V^2(r)\left(dx^0\right)^2 - W^2(r)\left(d\mathbf{x} - \mathbf{K}dx^0\right)\left(d\mathbf{x} - \mathbf{K}dx^0\right),$$

$$\mathbf{K} = \boldsymbol{\Omega}\times\mathbf{x},$$

$$g^{00} = \frac{1}{V^2(r)}, \quad g^{0i} = \frac{K^i}{V^2(r)}, \quad G^{ij} = -\frac{\delta^{ij}}{W^2(r)}.$$

In isotropic spherical coordinates ($\boldsymbol{\Omega}=\Omega\mathbf{e}_z$),

$$ds^2 = \left[V^2(r) - W^2(r)\Omega^2(r)r^2\sin^2\theta\right]\left(dx^0\right)^2$$
$$-W^2(r)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2 + 2\Omega(r)\sin^2\theta dx^0 d\phi\right)\right].$$

Exact Foldy-Wouthyusen Hamiltonian

$$T = m^2V^2 + F\mathbf{p}^2F - \frac{1}{4}\nabla F\cdot\nabla F + \frac{V}{2W^2}\left[F\left(\frac{2W'_r}{r} + W''_{rr}\right) + \frac{2V'_r}{r} + V''_{rr}\right] - \frac{1}{6}V^2R,$$

$$-\frac{1}{6}V^2R = \frac{1}{6}F\Delta F - \frac{V}{2W^2}\left[F\left(\frac{2W'_r}{r} + W''_{rr}\right) + \frac{2V'_r}{r} + V''_{rr}\right] + \frac{1}{12}\Omega'^2_r\, r^2\sin^2\theta,$$

$$\mathcal{H}_{FW} = \rho_3\sqrt{m^2V^2 + F\mathbf{p}^2F - \frac{1}{4}\nabla F\cdot\nabla F + \frac{1}{6}F\Delta F + \frac{1}{12}\Omega'^2_r\, r^2\sin^2\theta + \boldsymbol{\Omega}\cdot\mathbf{l}},$$

$\boldsymbol{\Omega} = -\mathbf{o}$ for rotating frame, $\boldsymbol{\Omega} = \frac{2G\mathbf{J}}{r^3}$ for Lense-Thirring metric,

$$\boldsymbol{\Omega} = \frac{2G\mathbf{J}}{r^3}\left[1 - \frac{3GM}{r} + \frac{21G^2M^2}{4r^2} + \mathcal{O}\left(\frac{a^2}{r^2}\right)\right]$$ for approximate Kerr metric,

$\mathbf{l} = \mathbf{r}\times\mathbf{p}$ is operator of angular momentum.

The Hamiltonian *is not changed* by the conformal–like transformation

Inclusion of electromagnetic interactions

Electromagnetic interactions can be added as follows:

$$\left[g^{\mu\nu}(\nabla_\mu + ieA_\mu)(\nabla_\nu + ieA_\nu) + m^2 - \lambda R\right]\psi = 0.$$

Simple derivation leads to

$$\left[\frac{1}{\sqrt{-g}}(\partial_\mu + ieA_\mu)\sqrt{-g}\,g^{\mu\nu}(\partial_\nu + ieA_\nu) + m^2 - \lambda R\right]\psi = 0.$$

Equivalent (relative to conformal-like transformations) form of this equation is given by

$$\left[\left(D_0 + \Upsilon'\right)^2 + D_i\frac{G^{ij}}{g^{00}}D_j + \Lambda + \frac{m^2}{g^{00}}\right]\Phi = 0,$$

Λ is the same

$$\Upsilon' = \frac{1}{2}\left\{D_i, \frac{g^{0i}}{g^{00}}\right\}, \qquad D_\mu = \partial_\mu + ieA_\mu.$$

Generalized Feshbach-Villars transformation for both massive and massless particles

$$\left[\left(\partial_0+\Upsilon'\right)^2+T'\right]\Phi=0,\qquad T'=D_i\frac{G^{ij}}{g^{00}}D_j+\Lambda+\frac{m^2}{g^{00}},$$

$$\Phi=\phi+\chi,\quad i\left(D_0+\Upsilon'\right)\Phi=N(\phi-\chi).$$

The nonunitary transformation results in the following generalized Feshbach-Villars Hamiltonian:

$$\mathcal{H}_{gFV}=\rho_3\frac{N^2+T'}{2N}+i\rho_2\frac{-N^2+T'}{2N}-i\Upsilon'.$$

This Hamiltonian *is not changed* by the conformal–like transformation $g'_{\mu\nu}=\Omega^2 g_{\mu\nu},\quad m'=\Omega^{-1}m.$

The considered equations *do not change* their conformal properties when electromagnetic interactions are included

Comparison of scalar and Dirac particles

Analysis of properties of conformal transformations for Dirac particles has been fulfilled in

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Scalar particle in general inertial and gravitational fields and conformal invariance revisited

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Conformal transformations for Dirac particles

gravity + electromagnetism

Covariant Dirac equation:

$$\left(i\hbar\gamma^a D_a - mc\right)\psi = 0, \quad D_a = e_a^\mu \partial_\mu + \frac{i}{4}\sigma^{bc}\Gamma_{bca}.$$

General nonunitary transformation brings this equation to the Hermitian Hamiltonian form:

$$\Phi = \sqrt{\sqrt{-g}\,e_{\hat{0}}^{0}}\,\psi.$$

Yu.N. Obukhov, A.J. Silenko, and O.V. Teryaev, Phys. Rev. D **84**, 024025 (2011).

The Hamiltonian *is not changed* by the conformal–like transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m.$$

Dirac and Foldy-Wouthuysen Hamiltonians are conformally invariant when *m*=0!

Wave function of the initial covariant Dirac equation possesses the following conformal property:

$$\psi' = \Omega^{3/2}\psi \quad \textbf{when} \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m.$$

Squared Dirac equation:

$$\left[\pi^i\pi_i - \frac{\hbar}{2}\sigma^{\alpha\beta}\left(\frac{q}{c}F_{\alpha\beta} + m\Phi_{\alpha\beta}\right) + \frac{\hbar^2}{4}R + \frac{\hbar^2}{16}(2\Gamma^i{}_{\alpha\beta}\Gamma_i{}^{\alpha\beta} + i\varepsilon^{\alpha\beta\mu\nu}\Gamma^i{}_{\alpha\beta}\Gamma_{i\mu\nu}\gamma_5) - m^2c^2\right]\psi = 0,$$

electromagnetic field tensor

For the squared Dirac equation, $\lambda = \frac{1}{4}$!

Summary

- The covariant Klein-Gordon equation is presented in a new (Hermitian) form and conformal symmetry for a massive pointlike scalar particle **(Higgs boson)** is found. Conformal transformations for a massless particle and conformal-like transformations for a massive one do not change the form of the obtained equation
- Generalized Feshbach-Villars transformation and Foldy-Wouthyusen are performed for both massive and massless scalar particles in arbitrary gravitational fields. Conformal symmetries of relativistic Hamiltonians are found in the general case.
- Exact Foldy-Wouthyusen transformations are fulfilled for an arbitrary static metric and for a frame rotating in the Kerr field approximated by a spatially isotropic metric
- It is proven that inclusion of electromagnetic interactions does not change conformal properties of the considered equations
- Conformal symmetries of equations for scalar and Dirac particles are very similar

Thank you for your attention