Noncommutative $SO(2,3)$ gauge theory and noncommutative gravity

V. Radovanović

University of Belgrade, Faculty of Physics

Beograd
August 2014

Outline

Introduction

1. M. Dimitrijević, V. Radovanović, Phys. Rev. D (2014)
2. M. Dimitrijević, V. Radovanović and H. Štefančić, Phys. Rev D (2012)

Introduction

Quantum field theory encounters problems at high energy/small distances.
There is no consistent (i. e. renormalizable and unitary) quantum theory of gravity.
Classical Relativity can not explain some phenomena in opservational cosmology. Both theories are incomplate.
Some modifications are needed.
String theory, Loop Quantum gravity, Extra dimensions ..
One possibility is noncommutativity among space time coordinates (Heisenberg, Pauli, Snyder). It is given by

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x) \ .$$

Introduction

Canonical noncommutativity

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} = \text{const.}$$

Different modes are constructed on canonical NC space time:
ϕ^4,QED, standard model, SUSY models;
renormalizability, unitary, phenomenological consequences, ...
Two important papers:
1. Seiberg and Witten, JHEP (1999)
2. Jurco, Moller, Schupp, Schraml and Wess, EPJ C, (2001)
Generalization general relativity or some other gravity theory (like Poincare gauge theory) on NC spacacetime is a difficult task.

Introduction

Many attempts:

• Twist approach: Commutative diffeomorphisms are replaced by twisted diffeomorphisms

1. P.Aschieri, C.Blohmann, M. Dimitrijević, F. Meyer, P.Schupp and J. Wess, CQG **2**2, 3511-3522 (2005),
2. P.Aschieri, M. Dimitrijević, F.Meyer and J.Wess, CQG **2**3, 1883-1912 (2006), [hep-th/0510059].

The physical meaning of twisted symmetry is unclear.

Introduction

- Sieberg-Witten approach

1. A. Chamseeddine, PLB. **5**04 (2001) 33;PRD. **6**9 (2004) 024015
2. M. A. Cardella and D. Zanon, CQG **2**0 (2003) L95
3. P. Aschieri and L. Castellani, JHEP (0906) (2009) 086
4. P. Aschieri and L. Castellani, arXiv:1111.4822, ArXiv: 12051911
5. R. Banerjee, P. Mukherjee, S. Samanta, PRD **7**5, (2007) 125020
6. Y. G. Miao, Z.Xue, S. J. Zhang, PRD **8**3, (2011) 024023

Gauging $GL(2,C)_\star$, $U(2,2)_\star, \dots$, breaking the symmetry down to Lorentz group.

$S^{(1)}_{NC} = 0$

AdS gauge theory on commutative spacetime

Consider a gauge theory with $SO(2,3)$ as a gauge group in $4D$ Minkowski spacetime.
$SO(2,3)$ is the isometry group anti de Sitter space.
Anti de Sitter space is a maximally symmetric space with a negative constant curvature.
M_{AB}-generators of $SO(2,3)$ group
$A, B, \ldots$ take values $0,1,2,3,5$.
Commutation relations:

$$[M_{AB}, M_{CD}] = i(\eta_{AD} M_{BC} + \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC}), \quad (1)$$

$\eta_{AB} = \mathrm{diag}(+,-,-,-,+)$ is $5D$ metric.

AdS gauge theory on commutative spacetime

Clifford generators Γ_A in $5D$ Minkowski space satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \ . \tag{2}$$

M_{AB} are

$$M_{AB} = \frac{i}{2}[\Gamma_A, \Gamma_B] \ . \tag{3}$$

$\gamma_a, \ (a = 0, 1, 2, 3)$ are the gamma matrices in $4D$ Minkovski spacetime

The gamma matrices in $5D$ are

$$\Gamma_A = (i\gamma_a\gamma_5, \gamma_5) \ .$$

γ_5 is defined by $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

AdS gauge theory on commutative spacetime

It is easy to show that

$$\begin{aligned} M_{ab} &= \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab} \ , \\ M_{5a} &= \frac{i}{2}\gamma_a \ . \end{aligned} \tag{4}$$

If we introduce momenta $P_a = \frac{1}{l}M_{a5}$,where l is a constant with dimensions of length AdS algebra (1) becomes

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}) \\ [M_{ab}, P_c] &= i(\eta_{bc}P_a - \eta_{ac}P_b) \\ [P_a, P_b] &= -i\frac{1}{l^2}M_{ab} \ . \end{aligned} \tag{5}$$

In the limit $l \to \infty$ AdS algebra reduces usual Poincare algebra in $4D$ spacetime. (Wigner-Inonu contraction)

AdS gauge theory on commutative spacetime

ψ spinor matter field in the fundamental representation
Under the infinitesimal $SO(2,3)$ gauge transformations it transforms as

$$\delta_\epsilon \psi = i\epsilon\psi = \frac{i}{2}\epsilon^{AB} M_{AB}\psi \tag{6}$$

The covariant derivative in the fundamental representation

$$D_\mu \psi = \partial_\mu \psi - i\omega_\mu \psi \ , \tag{7}$$

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB} M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma^{ab} - \frac{1}{2}\omega_\mu^{a5}\gamma_a \tag{8}$$

is the $SO(2,3)$ gauge potential. Decomposition: ω_μ^{AB} *to* ω_μ^{ab}, ω_μ^{a5},
ω_μ^{ab} is a spin connection
$\omega_\mu^{a5} = \frac{1}{l} e_\mu^a$ are vielbeins (tetrads).

AdS gauge theory on commutative spacetime

The transformation law of the $SO(2,3)$ potential is given by

$$\delta_\epsilon \omega_\mu^{AB} = \partial_\mu \epsilon^{AB} - \epsilon^A{}_C \omega_\mu^{CB} + \epsilon^B{}_C \omega_\mu^{CA} \ . \tag{9}$$

If $\epsilon^{a5} = 0$ we obtain the transformation laws for spin connection and vielbeins under the $SO(1,3)$ gauge transformation:

$$\delta_\epsilon \omega_\mu^{ab} = \partial_\mu \epsilon^{ab} - \epsilon^a{}_c \omega_\mu^{cb} + \epsilon^b{}_a \omega_\mu^{ca}, \tag{10}$$

$$\delta_\epsilon e_\mu^a = -\epsilon^{ad} e_\mu^d \tag{11}$$

We reduce the local anti de Sitter symmetry down to the local Lorentz symmetry:

$$SO(2,3) \rightarrow SO(1,3)$$

AdS gauge theory on commutative spacetime

The field strength

$$F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu,\omega_\nu] = \frac{1}{2}F_{\mu\nu}^{AB}M_{AB}$$
$$= \frac{1}{2}F_{\mu\nu}^{ab}M_{ab} + F_{\mu\nu}^{a5}M_{a5}\ , \qquad (12)$$

where

$$F_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - \frac{1}{l^2}(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a)\ . \qquad (13)$$

Reiman curvature tensor is

$$R_{\mu\nu}^{ab} = \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + \omega_\mu^{ac}\omega_\nu^{cb} - \omega_\mu^{bc}\omega_\nu^{ca} \qquad (14)$$

Torsion

$$lF_{\mu\nu}^{a5} = D_\mu e_\nu^a - D_\nu e_\mu^a = T_{\mu\nu}^a \qquad (15)$$

AdS gauge theory on commutative spacetime

Under the local anti de Sitter transformation field strength transforms as

$$\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}] \tag{16}$$

or more explicitly

$$\begin{aligned} \delta_\epsilon F_{\mu\nu}^{ab} &= -\epsilon^{ac} F_{\mu\nu c}{}^{b} + \epsilon^{bc} F_{\mu\nu c}{}^{a} - \epsilon^{a5} F_{\mu\nu 5}{}^{b} + \epsilon^{b5} F_{\mu\nu 5}{}^{a} \\ \delta_\epsilon T_{\mu\nu}^{a} &= -\epsilon^{ac} T_{\mu\nu c} + \epsilon^{5c} F_{\mu\nu c}{}^{a} \ . \end{aligned} \tag{17}$$

If we reduce the initial anti de Sitter gauge symmetry down to its Lorentz subgroup by taking $\epsilon^{a5} = 0$ we obtain the correct transformation laws:

$$\begin{aligned} \delta_\epsilon F_{\mu\nu}^{ab} &= -\epsilon^{a}{}_{c} F_{\mu\nu}^{cb} + \epsilon^{b}{}_{c} F_{\mu\nu}^{ca} \\ \delta_\epsilon T_{\mu\nu}^{a} &= -\epsilon^{a}{}_{c} T_{\mu\nu}^{c} \ . \end{aligned} \tag{18}$$

AdS gauge theory on commutative spacetime

Action:

$$S = \frac{il}{64\pi G_N}\int \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(F_{\mu\nu}F_{\rho\sigma}\phi\right) + \lambda\int d^4x\Big(\frac{1}{4}\mathrm{Tr}\,\phi^2 - l^2\Big) \quad (19)$$

is invariant under the $SO(2,3)$ gauge transformations
G_N is the Newton gravitational constant.
λ is the Lagrange multiplier.
$\phi = \phi^A\Gamma_A$ is a vector field

$$\delta\phi = i[\epsilon,\phi]\ , \quad (20)$$

Constraint $\phi_A\phi^A = l^2$.

AdS gauge theory on commutative spacetime

Taking $\phi^a = 0,\ \phi^5 = l$ the $SO(2,3)$ symmetry is broken down to $SO(1,3)$
The action after SB:

$$S = \frac{1}{16\pi G_N}\int d^4x\Big[\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{ab}R_{\rho\sigma}^{cd} + \sqrt{-g}R - 2\sqrt{-g}\Lambda\Big]\ , \quad (21)$$

where $\Lambda = -3/l^2$ and $\sqrt{-g} = \det e_\mu^a$.
MacDowell Mansurri action

AdS gauge theory on commutative spacetime

In this action (see the third line) the vielbeins and spin connection are independent variables. Varying the action with respect to the spin connection we obtain an equation which relates connection and vielbeins. In this way we can express the spin connection in terms of vielbeins. Since there is no fermionic matter in the action (21) this equation gives the vanishing torsion. In that case the first term in (21) is the Gauss-Bonnet term; it is a topological term and does not contribute to the equations of motion. The second term is the Einstein-Hilbert action, while the last term is the cosmological constant.

AdS gauge theory on commutative spacetime

From vielbeins we can construct the metric tensor

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \ . \tag{22}$$

The metric

$$g'_{\mu\nu} = \eta_{AB} D_\mu \Phi^A D_\nu \Phi^B \tag{23}$$

in the 'gauge' $\Phi^5 = l, \Phi^a = 0$ becomes $g_{\mu\nu}$. The action (21) is invariant under the Lorentz gauge transformations by construction. In addition this action posses invariance under general coordinate transformations. This action will be our starting point for the construction of a noncommutative gravity theory.

References:

MacDowell Mansurri, PRL (1977)
Stelle, West,PRD, 1980
P. Townsend, PRD, 1977

AdS gauge theory on commutative spacetime

Recently, a lot of attention has been devoted to this approach to Gravity:

1. A. Chamseddine and V. Mukhanov, 'Who Ordered the Anti-de Sitter Tangent Groups' Arxiv 1308.3199
2. J. Barret and S. Kerr, 'Gauge gravity and discrete quantum models' Arxiv: 1309.1660

Noncommutative $SO(2,3)_\star$ symmetry

Replace the usual product by the Moyal-Weyl $\star$ product

$$\hat{f}(\hat{x}) \cdot \hat{g}(\hat{x}) \to f \star g(x) \tag{24}$$

$$f(x) \star g(x) = e^{\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial y^\nu}} f(x) g(y)|_{y\to x} \ , \tag{25}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix,
$\star$-product is associative, noncommutative, reality condition.

Noncommutative $SO(2,3)_\star$ symmetry

Replace the commutative fields by their noncommutative counterparts.
Noncommutative gauge potential $\hat{\omega}_\mu$
Noncommutative curvature tensor $\hat{F}_{\mu\nu}$

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \stackrel{\star}{,} \hat{\omega}_\nu] \ . \tag{26}$$

Noncommutative adjoint field $\hat{\Phi}$

Noncommutative $SO(2,3)_\star$ symmetry

Under the deformed gauge transformations the gauge potential, the field strength and $\hat{\Phi}$ field transform as

$$\begin{aligned}
\delta^\star_\epsilon \hat{\omega}_\mu &= \partial_\mu \hat{\Lambda}_\epsilon - i[\hat{\omega}_\mu \overset{\star}{,} \hat{\Lambda}_\epsilon] \\
\delta^\star_\epsilon \hat{F}_{\mu\nu} &= i[\hat{\Lambda}_\epsilon \overset{\star}{,} \hat{F}_{\mu\nu}] \\
\delta^\star_\epsilon \hat{\Phi} &= i[\hat{\Lambda}_\epsilon \overset{\star}{,} \hat{F}]
\end{aligned} \tag{27}$$

where $\hat{\Lambda}_\epsilon$ is the noncommutative gauge parameter.
The noncommutative fields belong to the enveloping algebra of $so(2,3)$. For example, the $\star$-commutator in does not close in the Lie algebra.

Noncommutative $SO(2,3)_\star$ symmetry

Commutative and noncommutative symmetries which correspond to the same gauge group can be related by the Seiberg-Witten map: the map enables one to express the noncommutative variables in terms of the commutative variables. In that way no new degrees of freedom are introduced. SW map can also be seen as an expansion in $\theta^{\mu\nu}$, so the SW approach is known as a $\theta-$expanded theory. The noncommutative quantities $\hat{\Lambda}_\epsilon, \hat{\omega}_\mu, \hat{\Phi}$ are power series in the noncommutative parameter $\theta^{\mu\nu}$:

$$\begin{aligned} \hat{\Lambda}_\epsilon &= \epsilon + \hat{\Lambda}^{(1)} + \hat{\Lambda}^{(2)} + \dots \ , \\ \hat{\omega}_\mu &= \omega_\mu + \hat{\omega}_\mu^{(1)} + \hat{\omega}_\mu^{(2)} + \dots \ , \end{aligned} \tag{28}$$

where the higher order corrections are functions of the commutative variables ϵ, ω_μ, and their derivatives.

Noncommutative $SO(2,3)_\star$ symmetry

The requirement that the commutator of two deformed gauge transformations is a deformed transformation again:

$$[\delta^\star_\alpha \overset{\star}{,} \delta^\star_\beta] = \delta^\star_{-i[\alpha,\beta]} \tag{29}$$

gives the solution for $\Lambda^{(1)}_\epsilon, \Lambda^{(2)}_\epsilon, \dots$. The recursive relation between nth and $(n+1)$st order is

$$\hat{\Lambda}^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \overset{\star}{,} \partial_\lambda \hat{\epsilon}\}\Big)^{(n)} \ , \tag{30}$$

Noncommutative $SO(2,3)_\star$ symmetry

Solving the equation

$$\hat{\omega}_\mu(\omega) + \delta^\star_\epsilon \hat{\omega}_\mu(\omega) = \hat{\omega}_\mu(\omega + \delta_\epsilon \omega) \tag{31}$$

order by order in the noncommutative parameter we can express noncommutative gauge potential $\hat{\omega}_\mu$ in terms of the commutative one. The first order solution is

$$\hat{\omega}^{(1)}_\mu = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda \omega_\mu + F_{\lambda\mu}\} \tag{32}$$

Recursive relation:

$$\hat{\omega}^{(n+1)}_\mu = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \stackrel{\star}{,} \partial_\lambda \hat{\omega}_\mu + \hat{F}_{\lambda\mu}\}\Big)^{(n)}.$$

Noncommutative $SO(2,3)_\star$ symmetry

From potential $\hat{\omega}_\mu$ we find the field strength

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \stackrel{\star}{,} \hat{\omega}_\nu] \ , \tag{33}$$

The first order correction is

$$\hat{F}^{(1)}_{\mu\nu} = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda F_{\mu\nu} + D_\lambda F_{\mu\nu}\} + \frac{1}{2}\theta^{\kappa\lambda}\{F_{\mu\kappa}, F_{\nu\lambda}\} \tag{34}$$

Recursive relation:

$$\begin{aligned} \hat{F}^{(n+1)}_{\mu\nu} \ &= \ -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \stackrel{\star}{,} \partial_\lambda \hat{F}_{\mu\nu} + D_\lambda \hat{F}_{\mu\nu}\}\Big)^{(n)} \\ &\quad + \frac{1}{2(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{F}_{\mu\kappa}, \stackrel{\star}{,} \hat{F}_{\nu\lambda}\}\Big)^{(n)} \ . \end{aligned} \tag{35}$$

Noncommutative $SO(2,3)_\star$ symmetry

Transformation low

$$\delta^\star_\epsilon \hat{F}_{\mu\nu} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{F}_{\mu\nu}] \tag{36}$$

The field $\hat{\Phi}$ transforms as

$$\delta^\star_\epsilon \hat{\phi} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\phi}] \ . \tag{37}$$

Using the previous results we find the recursive relation

$$\hat{\phi}^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \stackrel{\star}{,} \partial_\lambda\hat{\phi} + D_\lambda\hat{\phi}\}\Big)^{(n)} \ , \tag{38}$$

Action

The NC action is given by

$$S_{NC} = -\frac{il}{16\pi G_N}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu}\star\hat{F}_{\rho\sigma}\star\hat{\phi}\,. \tag{39}$$

The $\star$-product is the Moyal-Weyl $\star$-product;
fields with "hat" are NC fields;
The action is invariant under the NC $SO(2,3)_\star$ gauge group

$$\delta^\star_\epsilon S_{NC} = \int \mathrm{d}^4x\partial_\mu K^\mu = \oint \mathrm{d}\Sigma_\mu K^\mu \tag{40}$$

Action

Seiberg-Witten expansion

$$S_{NC} = S^{(0)} + S^{(1)} + S^{(2)} + \dots \, . \tag{41}$$

Undre NC gauge transformation the 'field' $(\hat{F}_{\alpha\beta} \star \hat{F}_{\mu\nu})^{(1)}$ transforms in adjoint representation

$$\begin{aligned}(\hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma})^{(1)} &= F^{(1)}_{\mu\nu} F_{\rho\sigma} + F_{\mu\nu} F^{(1)}_{\rho\sigma} + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha F_{\mu\nu}\partial_\beta F_{\rho\sigma} \\ &= -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta(F_{\mu\nu}F_{\rho\sigma}) + D_\beta(F_{\mu\nu}F_{\rho\sigma})\} \\ &\quad + \frac{i}{2}\theta^{\alpha\beta}(D_\alpha F_{\mu\nu})(D_\beta F_{\rho\sigma}) + \frac{1}{2}\theta^{\alpha\beta}(\{F_{\alpha\mu}, F_{\beta\nu}\}F_{\rho\sigma} \\ &\quad + F_{\mu\nu}\{F_{\alpha\rho}, F_{\beta\sigma}\})\end{aligned}$$

The first term is simillar as the first term in (34), but

$$F \to FF \, . \tag{42}$$

Action

$$
\begin{aligned}
& (\hat{F}_{\alpha\beta} \star \hat{F}_{\mu\nu} \star \hat{\phi})^{(1)} \\
= \; & (F_{\alpha\beta} \star F_{\mu\nu})^{(1)} \phi + (F_{\alpha\beta} F_{\mu\nu}) \phi^{(1)} + \frac{i}{2} \partial_\alpha (F_{\alpha\beta} F_{\mu\nu}) \partial_\beta \phi \\
= \; & -\frac{1}{4} \theta^{\alpha\beta} \{\omega_\alpha, (\partial_\beta + D_\beta)(F_{\alpha\beta} F_{\mu\nu} \phi)\} + \frac{i}{2} \theta^{\alpha\beta} D_\alpha (F_{\mu\nu} F_{\rho\sigma}) D_\beta \phi \\
+ \; & \frac{i}{2} \theta^{\alpha\beta} D_\alpha F_{\mu\nu} D_\beta F_{\rho\sigma} \phi \\
+ \; & \frac{1}{2} \theta^{\alpha\beta} \{F_{\alpha\mu}, F_{\beta\nu}\} F_{\rho\sigma} \phi + \frac{1}{2} \theta^{\alpha\beta} F_{\mu\nu} \{F_{\alpha\rho}, F_{\beta\sigma}\} \phi \qquad (43)
\end{aligned}
$$

Action

The NC action in first order in θ is

$$\begin{aligned} S_{NC}^{(1)} &= \frac{il}{16\pi G_N}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}(\hat{F}_{\mu\nu}\star\hat{F}_{\rho\sigma}\star\hat{\phi})^{(1)} \\ &= \frac{il}{16\pi G_N}\theta^{\alpha\beta}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\Big(-\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}\{F_{\alpha\beta},\phi\} \\ &+ \frac{i}{2}D_\alpha F_{\mu\nu}D_\beta F_{\rho\sigma}\phi \\ &+ \frac{1}{2}\{F_{\alpha\mu},F_{\beta\nu}\}F_{\rho\sigma}\phi + \frac{1}{2}F_{\mu\nu}\{F_{\alpha\rho},F_{\beta\sigma}\}\phi\Big) \end{aligned} \tag{44}$$

Result $S_{NC}^{(1)} = 0$.

Action

From the first order of action, $S_{NC}^{(1)}$ we can easily find the second order.

$$
\begin{aligned}
S_{NC}^{(2)} &= \frac{il}{128\pi G_N}\theta^{\alpha\beta}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\Big(-\frac{1}{4}\hat{F}_{\mu\nu}\star\hat{F}_{\rho\sigma}\star\{\hat{F}_{\alpha\beta}\stackrel{\star}{,}\hat{\phi}\}\\
&+ \frac{i}{2}D_\alpha\hat{F}_{\mu\nu}\star D_\beta\hat{F}_{\rho\sigma}\star\hat{\phi}\\
&+ \frac{1}{2}\{\hat{F}_{\alpha\mu}\stackrel{\star}{,}\hat{F}_{\beta\nu}\}\star\hat{F}_{\rho\sigma}\star\hat{\phi}+\frac{1}{2}\hat{F}_{\mu\nu}\star\{\hat{F}_{\alpha\rho}\stackrel{\star}{,}\hat{F}_{\beta\sigma}\}\star\hat{\phi}\Big)^{(1)}
\end{aligned}
$$

Action

$$
\begin{aligned}
S_{NC}^{(2)} = \ & \frac{il}{64\pi G_N}\frac{1}{8}\theta^{\alpha\beta}\theta^{\kappa\lambda}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\Big\{\frac{1}{8}\{F_{\alpha\beta},\{F_{\mu\nu},F_{\rho\sigma}\}\}\{\phi,F_{\kappa\lambda}\} \\
& -\frac{1}{2}\{F_{\alpha\beta},\{F_{\rho\sigma},\{F_{\kappa\mu},F_{\lambda\nu}\}\}\}\phi-\frac{1}{4}\{\{F_{\mu\nu},F_{\rho\sigma}\},\{F_{\kappa\alpha},F_{\lambda\beta}\}\}\phi \\
& -\frac{i}{4}\{F_{\alpha\beta},[D_\kappa F_{\mu\nu},D_\lambda F_{\rho\sigma}]\}\phi-\frac{i}{2}[\{D_\kappa F_{\mu\nu},F_{\rho\sigma}\},D_\lambda F_{\alpha\beta}]\phi \\
& -\frac{1}{2}\{F_{\rho\sigma},\{F_{\alpha\mu},F_{\beta\nu}\}\}\{\phi,F_{\kappa\lambda}\}+\{\{F_{\alpha\mu},F_{\beta\nu}\},\{F_{\kappa\rho},F_{\lambda\sigma}\}\}\phi \\
& +2\{F_{\rho\sigma},\{F_{\beta\nu},\{F_{\kappa\alpha},F_{\lambda\mu}\}\}\}\phi+i\{F_{\rho\sigma},[D_\kappa F_{\alpha\mu},D_\lambda F_{\beta\nu}]\}\phi \\
& +2i[\{F_{\beta\nu},D_\kappa F_{\alpha\mu}\},D_\lambda F_{\rho\sigma}]\phi \\
& -\frac{i}{4}\{\phi,F_{\kappa\lambda}\}[D_\alpha F_{\mu\nu},D_\beta F_{\rho\sigma}]-\frac{1}{2}\{D_\kappa D_\alpha F_{\mu\nu},D_\lambda D_\beta F_{\rho\sigma}\}\phi \\
& +i[\{F_{\kappa\alpha},D_\lambda F_{\mu\nu}\},D_\beta F_{\rho\sigma}]\phi+i[\{F_{\lambda\nu},D_\alpha F_{\kappa\mu}\},D_\beta F_{\rho\sigma}]\phi \\
& +i[\{F_{\kappa\mu},D_\alpha F_{\lambda\nu}\},D_\beta F_{\rho\sigma}]\phi\Big\}\ .
\end{aligned}
$$

Action

The second order correction $S_{NC}^{(2)}$ is invariant under $SO(2,3)$. After the symmetry breaking the field $\phi^a = 0,\ \phi^5 = l$ we get

$$
\begin{aligned}
S_{NC}^{(2)} \;=\; & -\frac{l^2}{64\pi G_N}\theta^{\alpha\beta}\theta^{\kappa\lambda}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\int \mathrm{d}^4x\Big\{\frac{1}{256}\Big(F_{\mu\nu}{}^{cd}F_{\rho\sigma}{}^{ab}F_{\alpha\beta}{}^{mn}F_{\kappa\lambda mn} \\
& -8F_{\mu\nu}{}^{ab}F_{\rho\sigma}{}^{c5}F_{\kappa\lambda}{}^{de}F_{\alpha\beta e}{}^{5} + F_{\alpha\beta}{}^{ab}F_{\kappa\lambda}{}^{cd}(F_{\mu\nu}{}^{mn}F_{\rho\sigma mn} + 2F_{\mu\nu}{}^{m5}F_{\rho\sigma m}{}^{5})\Big) \\
& -\frac{1}{32}\Big(F_{\kappa\lambda}{}^{ab}F_{\mu\nu}{}^{cd}F_{\alpha\rho}{}^{mn}F_{\beta\sigma mn} + 2F_{\alpha\beta}{}^{ab}F_{\rho\sigma}{}^{cd}F_{\kappa\mu}{}^{m5}F_{\lambda\nu m}{}^{5} \\
& +F_{\kappa\mu}{}^{ab}F_{\lambda\nu}{}^{cd}F_{\alpha\beta}{}^{mn}F_{\rho\sigma mn}\Big) - \frac{1}{128}\Big(F_{\kappa\alpha}{}^{ab}F_{\lambda\beta}{}^{cd}(F_{\mu\nu}{}^{mn}F_{\rho\sigma mn} \\
& +2F_{\mu\nu}{}^{m5}F_{\rho\sigma m}{}^{5}) + F_{\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}(F_{\kappa\alpha}{}^{mn}F_{\lambda\beta mn} + 2F_{\kappa\alpha}{}^{m5}F_{\lambda\beta m}{}^{5})\Big) \qquad (45)
\end{aligned}
$$

Action

$$
\begin{aligned}
&+\frac{1}{16}F_{\alpha\beta}{}^{ab}\Big(\big((D_\kappa F_{\mu\nu})^{cm}\big)(D_\lambda F_{\rho\sigma})_m^{\ d} + (D_\kappa F_{\mu\nu})^{c5}(D_\lambda F_{\rho\sigma})_5{}^{d}\Big)\\
&-\frac{1}{16}\Big((D_\kappa F_{\mu\nu})^{ab}(D_\lambda F_{\alpha\beta})^{d5}F_{\rho\sigma}^{c5} + (D_\kappa F_{\mu\nu})^{a5}(D_\lambda F_{\alpha\beta})^{b5}F_{\rho\sigma}{}^{cd}\Big)\\
&+\frac{1}{16}F_{\alpha\mu}^{ab}F_{\beta\nu}^{cd}\Big(F_{\kappa\rho}{}^{mn}F_{\lambda\sigma mn} + 2F_{\kappa\rho}{}^{m5}F_{\lambda\sigma m5}\Big)\\
&+\frac{1}{16}\Big(F_{\rho\sigma}{}^{ab}F_{\beta\nu}{}^{cd}(F_{\kappa\alpha}{}^{mn}F_{\lambda\mu mn} + 2F_{\kappa\alpha}{}^{m5}F_{\lambda\mu m5}) + F_{\kappa\alpha}{}^{ab}F_{\lambda\mu}{}^{cd}F_{\rho\sigma}{}^{mn}F_{\beta\nu mn}\\
&-4(F_{\kappa\alpha}{}^{ab}F_{\lambda\mu}{}^{c5} + F_{\kappa\alpha}{}^{a5}F_{\lambda\mu}{}^{bc})F_{\rho\sigma}{}^{de}F_{\beta\nu e5}\Big)\\
&-\frac{1}{8}F_{\rho\sigma}{}^{ab}\Big((D_\kappa F_{\alpha\mu})^{cm}(D_\lambda F_{\beta\nu})_m^{\ d} + (D_\kappa F_{\alpha\mu})^{c5}(D_\lambda F_{\beta\nu})_5{}^{d}\Big)\\
&+\frac{1}{2}\Big(F_{\kappa\mu}{}^{ab}(D_\alpha F_{\lambda\nu})^{c5}(D_\beta F_{\rho\sigma})^{d5} + F_{\kappa\mu}{}^{a5}(D_\alpha F_{\lambda\nu})^{bc}\Big)(D_\beta F_{\rho\sigma})^{d5}\\
&+\frac{1}{8}\Big(F_{\kappa\alpha}{}^{ab}(D_\lambda F_{\mu\nu})^{c5} + F_{\kappa\alpha}{}^{a5}(D_\lambda F_{\mu\nu})^{bc}\Big)(D_\beta F_{\rho\sigma})^{d5}
\end{aligned}
$$

Action

$$
\begin{aligned}
(D_\alpha F_{\mu\nu})^{ab} &= \nabla_\alpha F_{\mu\nu}{}^{ab} - \frac{1}{l^2}(e_\alpha^a T_{\mu\nu}{}^b - e_\alpha^b T_{\mu\nu}{}^a) \\
(D_\alpha F_{\mu\nu})^{a5} &= \frac{1}{l}(\nabla_\alpha T_{\mu\nu}^a + e_\alpha^m F_{\mu\nu m}{}^a)\,, \\
(D_\kappa D_\alpha F_{\mu\nu})^{ab} &= \nabla_\kappa \nabla_\alpha F_{\mu\nu}{}^{ab} - \frac{1}{l^2}\Big((\nabla_\kappa e_\alpha^a) T_{\mu\nu}{}^b + \dots\Big)\,,
\end{aligned}
$$

with the $SO(1,3)$ covariant derivative
$\nabla_\alpha F_{\mu\nu}{}^{ab} = \partial_\alpha F_{\mu\nu}{}^{ab} + \omega_\alpha^{ac} F_{\mu\nu c}{}^b - \omega_\alpha^{bc} F_{\mu\nu c}{}^a$ and
$\nabla_\alpha T_{\mu\nu}^a = \partial_\alpha T_{\mu\nu}^a + \omega_\alpha^{ac} T_{\mu\nu c}$.

Action

Three scales in the model: $\Lambda = -\frac{3}{l^2}$, θ, R^n. Terms in the action ($T = 0$)

$$\frac{1}{l^2 G_N}\left\{1, Rl^2, \frac{\theta^2}{l^4}\Big(1,\ Rl^2,\ R^2 l^4,\ R^3 l^6,\ R^4 l^8\Big)\right\}.$$

Expansion 1: big cosmological constant, smaller energy

Action

$$
\begin{aligned}
S^{(2)} &= \frac{3\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N l^6}\int \mathrm{d}^4x\sqrt{-g}\,g_{\alpha\kappa}g_{\beta\lambda} \\
&-\frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N l^4}\int \mathrm{d}^4x\sqrt{-g}\Big(3g_{\alpha\kappa}R_{\beta\lambda}+3R_{\alpha\beta\kappa\lambda}-2R_{\alpha\kappa\beta\lambda}\Big) \\
&+\frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{256\pi G_N l^4}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e_\lambda{}^c e_\sigma{}^d\nabla_\kappa\nabla_\alpha(l^2R_{\mu\nu}{}^{ab}-2e_\mu{}^a e_\nu{}^b)g_{\beta\rho} \\
&-\frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{256\pi G_N l^2}\int \mathrm{d}^4x\sqrt{-g}\Big(2g_{\alpha\kappa}(-2RR_{\beta\lambda} \\
&+4R_{\beta\mu}R^{\mu}_{\ \lambda}+4R^{\mu\nu}R_{\beta\nu\lambda\mu}-2R_{\beta\mu}{}^{\rho\sigma}R_{\rho\sigma\lambda}{}^{\mu}) \\
&+R(2R_{\alpha\kappa\beta\lambda}+R_{\alpha\beta\kappa\lambda})+2R_{\alpha\kappa}R_{\beta\lambda}-16R_{\alpha\beta\kappa\mu}R^{\mu}_{\ \lambda}-16R_{\alpha\kappa\beta\mu}R_{\lambda}{}^{\mu}
\end{aligned}
$$

Action

$$-2R_{\alpha\beta}{}^{\mu\nu}(R_{\kappa\lambda\mu\nu}-6R_{\kappa\mu\lambda\nu})+2R_{\alpha\kappa}{}^{\mu\nu}(5R_{\beta\lambda\mu\nu}-4R_{\beta\mu\lambda\nu})$$

$$+4R_{\alpha\mu\beta}{}^{\nu}R_{\kappa\nu\lambda}{}^{\mu}+4R_{\alpha\mu\kappa}{}^{\nu}R_{\lambda\nu\beta}{}^{\mu}-6R_{\alpha\mu\kappa}{}^{\nu}R_{\beta\nu\lambda}{}^{\mu}\Big)$$

$$+\frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{256\pi G_N l^2}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big\{-2e_\alpha{}^a e_\beta{}^b(\nabla_\kappa(R_{\mu\nu}{}^{cm})\nabla_\lambda(e_{\rho m}e_\sigma{}^d)$$

$$+\frac{2}{l^2}\nabla_\kappa(e_\mu{}^c e_\nu{}^m)\nabla_\lambda(e_{\rho m}e_\sigma{}^d))$$

$$+e_\rho{}^a e_\sigma{}^b\Big[2\nabla_\kappa R_{\alpha\mu}{}^{cm}\nabla_\lambda(e_{\beta m}e_\nu{}^d-e_\beta{}^d e_{\nu m})$$

$$-\frac{1}{l^2}\nabla_\kappa(e_\alpha{}^c e_\mu{}^m-e_\alpha{}^m e_\mu{}^c)\nabla_\lambda(e_{\beta m}e_\nu{}^d-e_\beta{}^d e_{\nu m})\Big]$$

$$+\frac{1}{2}\nabla_\kappa\nabla_\alpha(e_\mu{}^a e_\nu{}^b)\big(\nabla_\lambda\nabla_\beta(e_\rho{}^c e_\sigma{}^d)+2e_\lambda{}^c R_{\rho\sigma\beta}{}^{\omega}e_\omega{}^d\big)\Big\}$$

$$-\frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{256\pi G_N l^2}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big\{R_{\alpha\beta}{}^{ab}\nabla_\kappa(e_\mu{}^c e_\nu{}^m)\nabla_\lambda(e_{\rho m}e_\sigma{}^d)$$

Action

Action is not invariant under general coordinates transformation: θ is not a tensor;

$$\nabla_\mu e_\rho{}^a = \partial_\mu e_\rho{}^a + \omega_\mu^{ab} e_{\rho b} = \Gamma_{\mu\rho}^\sigma e_\sigma{}^a \ . \tag{46}$$

Theory is invariant under twisted diffeomorphism.
If $1 \gg l^2 R \gg \frac{\theta^2}{l^4}$

$$S^{(2)} = \frac{3\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N l^6} \int \mathrm{d}^4x \sqrt{-g}\, g_{\alpha\kappa} g_{\beta\lambda} \ . \tag{47}$$

$$\Lambda(x) = \Lambda - \frac{3}{8} \frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{l^6} g_{\alpha\kappa} g_{\beta\lambda} \ . \tag{48}$$

Action

Expansion 2: On higher energy the higher powers in R are important.

$$S^{(2)} = -\frac{l^2\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N}\int d^4xe\Big[R_{\mu\nu\gamma\delta}R^{\rho\sigma\gamma\delta}\Big(\frac{1}{32}(2R_{\alpha\beta}{}^{\mu}{}_{\rho}R_{\kappa\lambda}{}^{\nu}{}_{s} - R_{\alpha\beta}{}^{\mu\nu}R_{\kappa\lambda\rho\sigma})$$
$$+\frac{1}{16}R_{\kappa\alpha\rho\sigma}R_{\lambda\beta}{}^{\mu\nu} - \frac{1}{8}R_{\kappa\alpha\nu\sigma}R_{\lambda\beta}{}^{\mu}{}_{\rho}\Big)$$
$$+\frac{1}{4}R_{\kappa\lambda\gamma\delta}R^{\rho\sigma\gamma\delta}(R_{\alpha\mu\rho\sigma}R_{\beta}{}^{\mu} + R_{\alpha\mu\rho\nu}R_{\beta}{}^{\nu}{}_{\sigma}{}^{\mu} - R_{\alpha\rho}R_{\beta\sigma})$$
$$+\frac{1}{8}R_{\alpha\mu\gamma\delta}R_{\beta\nu}{}^{\gamma\delta}(RR_{\kappa\lambda}{}^{\mu\nu} + R_{\kappa\lambda\rho\sigma}R^{\rho\sigma\mu\nu} + 4R_{\kappa\lambda}{}^{\nu}{}_{\rho}R^{\rho\mu})$$
$$+\frac{1}{2}R_{\alpha\rho\gamma\delta}R_{\beta\sigma}{}^{\gamma\delta}(R_{\kappa\mu}{}^{\rho\sigma}R_{\lambda}{}^{\mu} + R_{\kappa\mu\nu\rho}R_{\lambda}{}^{\nu\mu\sigma} + R_{\kappa\sigma}R_{\lambda\rho})$$
$$-\frac{1}{4}R_{\kappa\alpha\gamma\delta}R_{\lambda\mu}{}^{\gamma\delta}(R_{\beta\nu\rho\sigma}R^{\rho\sigma\mu\nu} + RR_{\beta\mu} - 2R_{\sigma\mu}R_{\beta}{}^{\sigma} - 2R_{\sigma\nu}R_{\beta}{}^{\nu\mu\sigma}) + \dots \quad (49)$$

Action

Expansion 3: $R_{\mu\nu}{}^{ab}=0,\ T\neq 0$

$$
\begin{aligned}
S_T = &-\frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N l^6}\int \mathrm{d}^4x\sqrt{-g}\Big[5\,T^{m}_{\kappa\alpha}T_{\lambda\beta m}-\frac{11}{4}T^{m}_{\alpha\beta}T_{\kappa\lambda m}\\
&-\frac{3}{2}T^{\rho}_{\rho\kappa}T_{\alpha\beta\lambda}+T^{\nu}_{\alpha\lambda}T_{\beta\nu\kappa}-5\,T_{\alpha\kappa\beta}T^{\mu}_{\mu\lambda}+2\,T^{\mu}_{\alpha\beta}T_{\kappa\mu\lambda}\\
&+\frac{1}{2}g_{\lambda\nu}e^{\nu}_{c}\partial_\kappa T^{c}_{\alpha\beta}-g_{\lambda\alpha}e^{\nu}_{a}\partial_\kappa T^{a}_{\nu\beta}\\
-\ \ &\frac{1}{2}(e^{\mu}_{c}e^{\nu}_{d}-e^{\nu}_{c}e^{\mu}_{d})\partial_\kappa(e^{c}_{\alpha}e^{m}_{\mu}-e^{m}_{\alpha}e^{c}_{\mu})\partial_\lambda(e_{\beta m}e^{d}_{\nu}-e^{d}_{\beta}e_{\nu m})\\
&-\partial_\kappa(e^{c}_{\alpha}e^{m}_{\mu}-e^{m}_{\alpha}e^{c}_{\mu})\Big(e^{\mu}_{c}(e_{\lambda m}T^{\nu}_{\mu\nu}-T_{\beta\lambda m})-e^{\nu}_{c}(e_{\lambda m}T^{\mu}_{\beta\nu}-\delta^{\mu}_{\lambda}T_{\beta\nu m})\Big)\\
&+\dots
\end{aligned}
\tag{50}
$$

Action

We can start from commutative action

$$S = \frac{1}{64\pi G_N l}\int d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(F_{\mu\nu}D_\rho\Phi D_\sigma\Phi\Phi\right) \tag{51}$$

After SSB we get

$$S = -\frac{1}{16\pi G_N l}\int d^4xe(R - \frac{12}{l^2})\ . \tag{52}$$

Deformation of this term is in progress (with M. Dimitrijevic and B. Nikolic)

1. H. Pagels, Phys. Rev D (1984)
2. F. Wilczek, Phys. Rev. Lett (1998)

Conclusion

A gauge theory on Moyal space with $SO(2,3)_\star$ symmetry is constructed; We find SW expansion of the noncommutative action up to the second order in θ.

The commutative symmetry is broken

$$SO(2,3) \rightarrow SO(1,3) \tag{53}$$

In the lowest order we get EH term + cosmological constant term + GB term.

The second order corrections in θ are found in the covariant form .

Future investigations:

NC corrections to the black hole/ cosmological solutions

NC deformation of SUGR