Closed string noncommutativity in the weakly curved background

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8th School and Conference of Modern Mathematical Physics
24.-31. August 2014, Belgrade, Serbia

Outline of the talk

Open string noncommutativity

- Space-time looks different from the string point of view.
- Open string endpoints became noncommutative in the presence of the constant Kalb-Ramond field $B_{\mu\nu}$.
- Minimum action principle $\delta S = 0$

$$S(x) = \kappa \int_\Sigma d^2\xi \left(\frac{1}{2} \eta^{\alpha\beta} G_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu} \right) \partial_\alpha x^\mu \partial_\beta x^\nu \, ,$$

gives equations of motion and boundary conditions.

Open string noncommutativity

- Solution of the boundary conditions is of the form

$$x^\mu = q^\mu - \Theta^{\mu\nu} \int d\eta p_\nu(\eta)\,,$$

  where q^μ and p_ν are effective variables satisfying $\{q^\mu(\sigma), p_\nu(\bar\sigma)\} = \delta^\mu{}_\nu \delta_s(\sigma, \bar\sigma)$.
- Coordinate x^μ is the linear combination of the effective coordinate q^μ and effective momentum p_ν which produces the noncommutativity

$$\{x^\mu(0), x^\nu(0)\} = -2\Theta^{\mu\nu}\,, \quad \{x^\mu(\pi), x^\nu(\pi)\} = 2\Theta^{\mu\nu}\,.$$

Open string noncommutativity

- Effective action is of the form

$$S(q) = S(x)|_{bc} = \frac{1}{2}\int_\Sigma d^2\xi \eta^{\alpha\beta} G^E_{\mu\nu}\partial_\alpha q^\mu \partial_\beta q^\beta \,,$$

where

$$G^E_{\mu\nu} = (G - 4BG^{-1}B)_{\mu\nu}\,, \quad \Theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}\,,$$

are efective metric and noncommutativity parameter, respectively.

Closed string T-duality

- T-duality connects physically equivalent theories with different backgrounds.
- There are two main consequences of the compactification on a circle:
  - momentum bacames quantized - $p = \frac{n}{R}\,(n \in Z)$
  - new states appear (winding modes)

$$x(2\pi) - x(0) = 2\pi RN\,.$$

- Mass squered of any state

$$M^2 = \frac{n^2}{R^2} + m^2 \frac{R^2}{\alpha'^2}\,,$$

is invariant under exchanges $n \leftrightarrow m\,,\, R \leftrightarrow {}^\star R = \frac{\alpha'}{R}$.

Closed string T-duality

- Compactification on a circle of radius R is equivalent to the compactification on a circle of radius ${}^\star R$.
- T-dual action ${}^\star S$ is of the same form as the initial one but with background fields

$$ {}^\star G^{\mu\nu} \sim (G_E^{-1})^{\mu\nu}\,, \quad {}^\star B^{\mu\nu} \sim \Theta^{\mu\nu}\,. $$

Choice of the background fields

- $G_{\mu\nu} = const.$ and $B_{\mu\nu} = b_{\mu\nu} + h_{\mu\nu}(x) \equiv b_{\mu\nu} + \frac{1}{3}H_{\mu\nu\rho}x^\rho$.
- $b_{\mu\nu}$ and $H_{\mu\nu\rho}$ are constants and $H_{\mu\nu\rho}$ is infinitesimal.
- Background fields satisfy the space-time equations of motion (consistency conditions). Ricci tensor $R_{\mu\nu}$ is proportional to $H^2_{\mu\nu}$, so, in linear approximation in $H_{\mu\nu\rho}$ space-time metric can be considered as constant.

Generalized Buscher rules

- There are two steps in generalized Buscher procedure:
  - localization of global shift symmetry $\delta x^\mu = \lambda^\mu$

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha \, ,$$

  - $x^\mu \to \Delta x^\mu_{inv} = \int_P d\xi^\alpha D_\alpha x^\mu$. (new step)

T-dual action

- $x^\mu \to V^\mu = -\kappa\Theta_0^{\mu\nu} y_\nu + (g_E^{-1})^{\mu\nu}\tilde{y}_\nu$.
- ${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}(\Delta V)$ and ${}^\star B^{\mu\nu} = \frac{\kappa}{2}\Theta^{\mu\nu}(\Delta V)$.
- The T-dual action is of the form

$$ {}^\star S = \frac{\kappa^2}{2}\int_\Sigma d^2\xi\, \partial_+ y_\mu \Theta_-^{\mu\nu}(\Delta V)\partial_- y_\nu \,, $$

where

$$ \Theta_\pm^{\mu\nu}(x) = -\frac{2}{\kappa}(G_E^{-1}(x)\Pi_\pm(x)G^{-1})^{\mu\nu} \,, \quad \Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2}G_{\mu\nu} . $$

T-dual transformations in the canonical form

$$x'^{\mu} = \frac{1}{\kappa}{}^{\star}\pi^{\mu} - \kappa\Theta_0^{\mu\nu}\beta_\nu^0(V) - (g_E^{-1})^{\mu\nu}\beta_\nu^1(V)\,,$$

$$y'_\mu = \frac{1}{\kappa}\pi_\mu - \beta_\mu^0(x)\,.$$

Infinitesimally small terms (contain β_μ^α) are corrections with respect to the constant background case. These terms are the source of noncommutativity.

Calculation of Poisson brackets

$$\Delta X_\mu(\sigma,\sigma_0)=\int_{\sigma_0}^{\sigma} d\sigma_1 X'_\mu(\sigma_1)\,,\quad \Delta Y_\mu(\sigma,\sigma_0)=\int_{\sigma_0}^{\sigma} d\sigma_1 Y'_\mu(\sigma_1)\,.$$

From

$$\{X'_\mu(\sigma), Y'_\nu(\bar\sigma)\}=K'_{\mu\nu}(\sigma)\delta(\sigma-\bar\sigma)+L_{\mu\nu}(\sigma)\delta'(\sigma-\bar\sigma)\,,$$

we obtain

$$\{X_\mu(\sigma), Y_\nu(\bar\sigma)\}=-\left[K_{\mu\nu}(\sigma)-K_{\mu\nu}(\bar\sigma)+L_{\mu\nu}(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,.$$

Taking $\sigma\to 2\pi+\sigma$ and $\bar\sigma\to\sigma$ we get

$$\{X_\mu(2\pi+\sigma), Y_\nu(\sigma)\}=-\left[K_{\mu\nu}(2\pi+\sigma)-K_{\mu\nu}(\sigma)+L_{\mu\nu}(\sigma)\right]\,.$$

Origin of noncommutativity

Calculation of Poisson brackets is in fact a calculation of Poisson brackets of the coordinate sigma derivatives. Sigma derivatives of the coordinates are given by T-dual transformations in the canonical form. Since the T-dual transformation of the coordinate is given as linear combination of the coordinates and their canonically conjugated momenta from the T-dual space, we get the noncommutativity of the coordinates. For constant background case there is no noncommutativity

$$\pi_\mu = \kappa y'_\mu \,, \quad {}^\star\pi^\mu = \kappa x'^\mu \,,$$

$$\{\pi_\mu, \pi_\nu\} = 0 \Rightarrow \{y_\mu, y_\nu\} = 0 \,.$$

Definitions

- ${}^\star B^{\mu\nu}$ depends on $\Delta V(y, \tilde{y})$

$$ {}^\star B^{\mu\nu} = b^{\mu\nu} + Q^{\mu\nu}{}_\rho \Delta V^\rho \, . $$

- Christofel symbol for $G^E_{\mu\nu}$ is denoted by $\Gamma^E_{\mu,\nu\rho}$.
- $\tilde{y}'_\mu = \dot{y}_\mu$.

Results

$$\{y_\mu(2\pi+\sigma), y_\nu(\sigma)\} = -\frac{2\pi}{\kappa} H_{\mu\nu\rho} N^\rho \,,$$

$$\{y_\mu(2\pi+\sigma), \tilde{y}_\nu(\sigma)\} + \{y_\mu(\sigma), \tilde{y}_\nu(2\pi+\sigma)\}$$
$$= \quad -\frac{4\pi}{\kappa^2} H_{\mu\nu\rho} p^\rho + \frac{\pi}{\kappa}\left(3\Gamma^E_{\rho,\mu\nu} - 8H_{\mu\nu\lambda} b^\lambda{}_\rho\right) N^\rho \,,$$

$$\{\tilde{y}_\mu(2\pi+\sigma), \tilde{y}_\nu(\sigma)\} = \frac{2\pi}{\kappa}\left[-H_{\mu\nu\rho} - 6g_{\mu\alpha} Q^{\alpha\beta}{}_\rho g_{\beta\nu}\right] N^\rho$$
$$+ \quad \frac{2\pi}{\kappa}\left[2H_{\mu\nu}{}^\lambda g_{\lambda\rho} + 3\left(\Gamma^E_{\mu,\nu\lambda} - \Gamma^E_{\nu,\mu\lambda}\right) b^\lambda{}_\rho\right] N^\rho$$
$$+ \quad \frac{\pi}{\kappa^2}\left[3\left(\Gamma^E_{\mu,\nu\rho} - \Gamma^E_{\nu,\mu\rho}\right) - 8H_{\mu\nu\lambda} b^\lambda{}_\rho\right] p^\rho \,.$$

Winding numbers and momenta

$$N^{\mu} = \frac{1}{2\pi}\left[x^{\mu}(2\pi+\sigma) - x^{\mu}(\sigma)\right] , \quad {}^{\star}N_{\mu} = \frac{1}{2\pi}\left[y_{\mu}(2\pi+\sigma) - y_{\mu}(\sigma)\right] ,$$

$$p_{\mu} = \frac{1}{2\pi}\int_{\sigma}^{2\pi+\sigma} d\eta \pi_{\mu}(\eta) , \quad {}^{\star}p^{\mu} = \frac{1}{2\pi}\int_{\sigma}^{2\pi+\sigma} d\eta {}^{\star}\pi^{\mu}(\eta) .$$

Algebra of winding numbers and momenta

For calculation we need expressions calculated earlier and

$$\Delta y_\mu(2\pi,0) \quad = 2\pi^\star N_\mu \,, \quad \Delta \tilde{y}_\mu(2\pi,0) = 2\pi^\star P_\mu \,,$$

$$\Delta x^\mu = 2\pi N^\mu \,, \quad \Delta \tilde{x}^\mu = 2\pi P^\mu \,.$$

Algebra of winding numbers and momenta

$$\{{}^\star N_\mu, {}^\star N_\nu\} = \frac{1}{\pi\kappa} H_{\mu\nu\rho} N^\rho \,,$$

$$\{{}^\star N_\mu, {}^\star P_\nu\} = \frac{1}{\pi\kappa} H_{\mu\nu\rho} P^\rho - \frac{3}{4\pi\kappa} \Gamma^E_{\rho,\mu\nu} N^\rho \,,$$

$$\begin{aligned}
\{{}^\star P_\mu, {}^\star P_\nu\} &= -\frac{1}{\pi\kappa}\left(H_{\mu\nu\rho} - 6 g_{\mu\alpha} Q^{\alpha\beta}{}_\rho g_{\beta\nu}\right) N^\rho \\
&+ \frac{1}{\pi}\left[-\frac{3}{2\kappa}(\Gamma^E_{\mu,\nu\rho} - \Gamma^E_{\nu,\mu\rho}) + \frac{4}{\kappa} H_{\mu\nu\sigma}(G^{-1}b)^\sigma{}_\rho\right] P^\rho \,.
\end{aligned}$$

Concluding remarks

- If there is a nontrivial winding in the weakly curved background, using T-duality transformation laws we get closed string noncommutativity.
- Poisson brackets are, in general, proportional to the linear combination of the T-dual winding number and momenta modes.
- We obtain the algebra of the T-dual winding number and momenta.