Algebraic Bethe ansatz for the XXX chain with triangular boundaries and Gaudin model

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Yang R-matrix

The Heisenberg XXX spin chain is related to the Yangian $\mathcal{Y}(s\ell(2))$ and the $SL(2)$-invariant Yang R-matrix

$$R(\lambda) = \lambda \mathbb{1} + \eta \mathcal{P} = \begin{pmatrix} \lambda + \eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda + \eta \end{pmatrix},$$

where λ is a spectral parameter, η is a quasi-classical parameter. We use $\mathbb{1}$ for the identity matrix and $\mathcal{P}$ for the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Yang-Baxter equation

The Yang R-matrix satisfies the Yang-Baxter equation in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu),$$

here the standard notation is used to denote spaces $V_j, j = 1, 2, 3$ on which corresponding R-matrices $R_{ij}, ij = 12, 13, 23$ act non-trivially. Evidently, in the present case $V_1 = V_2 = V_3 = \mathbb{C}^2$.

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Properties of the Yang R-matrix

The Yang R-matrix also satisfies other relevant properties such as

- Unitarity $R_{12}(\lambda)R_{21}(-\lambda) = (\eta^2 - \lambda^2)\mathbb{1};$
- Parity invariance $R_{21}(\lambda) = R_{12}(\lambda);$
- Temporal invariance $R^t_{12}(\lambda) = R_{12}(\lambda);$
- Crossing symmetry $R(\lambda) = \mathcal{J}_1 R^{t_2}(-\lambda - \eta)\mathcal{J}_1^{-1},$
  where t_2 denotes the transpose in the second space and the entries of the two-by-two matrix $\mathcal{J}$ are $\mathcal{J}_{ab} = (-1)^{a-1}\delta_{a,3-b}$.

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Hilbert space

We study an inhomogeneous XXX spin chain with N sites, with the local space $V_m = \mathbb{C}^{2s+1}$ and inhomogeneous parameter α_j. For simplicity, we start by considering periodic boundary conditions. The Hilbert space of the system is

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}.$$

Lax operator

Following Sklyanin '87 we introduce the Lax operator

$$\mathbb{L}_{0m}(\lambda) = \mathbb{1} + \frac{\eta}{\lambda}\left(\vec{\sigma}_0 \cdot \vec{S}_m\right) = \frac{1}{\lambda}\begin{pmatrix} \lambda + \eta S_m^3 & \eta S_m^- \\ \eta S_m^+ & \lambda - \eta S_m^3 \end{pmatrix},$$

here

$$S_m^\alpha = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1},$$

with $m = 1, 2, \ldots, N$ and S^α, $\alpha = 1, 2, 3$ are the spin operators acting in a local $\mathbb{C}^{2s+1}$. Evidently, $\mathbb{L}(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$.

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Lax operator

It obeys

$$\mathbb{L}_{0m}(\lambda)\mathbb{L}_{0m}(\eta-\lambda) = \left(1+\eta^2\,\frac{s_m(s_m+1)}{\lambda(\eta-\lambda)}\right)\mathbb{1}_0\,,$$

where s_m is the value of spin in the space V_m.

Due to the usual commutation relations the spin operators S_m^α, it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda-\mu)\mathbb{L}_{0m}(\lambda-\alpha_m)\mathbb{L}_{0'm}(\mu-\alpha_m) = \mathbb{L}_{0'm}(\mu-\alpha_m)\mathbb{L}_{0m}(\lambda-\alpha_m)R_{00'}(\lambda-\mu).$$

Monodromy matrix

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N) \cdots \mathbb{L}_{01}(\lambda - \alpha_1)$$

is used to describe the system. Notice that $T(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

RTT-relations

From RLL-relations it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda-\mu)T_0(\lambda)T_{0'}(\mu)=T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda-\mu).$$

The RTT-relations define the commutation relations for the entries of the monodromy matrix.

Hilbert space

In every $V_m = \mathbb{C}^{2s+1}$ there exists a vector $\omega_m \in V_m$ such that

$$S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0.$$

We define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}.$$

Hilbert space

From the definitions above it is straightforward to obtain the action of the entries of the monodromy matrix $T(\lambda)$ on the vector Ω_+

$$
\begin{aligned}
A(\lambda)\Omega_+ &= a(\lambda)\Omega_+, \quad \text{with} \quad a(\lambda) = \prod_{m=1}^{N} \frac{\lambda - \alpha_m + \eta s_m}{\lambda - \alpha_m}, \\
D(\lambda)\Omega_+ &= d(\lambda)\Omega_+, \quad \text{with} \quad d(\lambda) = \prod_{m=1}^{N} \frac{\lambda - \alpha_m - \eta s_m}{\lambda - \alpha_m}, \\
C(\lambda)\Omega_+ &= 0.
\end{aligned}
$$

To construct integrable spin chains with non-periodic boundary condition, we will follow Sklyanin's approach.

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Reflection equation

A way to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk system, was developed by Sklyanin '88.

The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. It is written in the following form for the left reflection matrix acting on the space $V_1 = \mathbb{C}^2$ at the first site, $K^-(\lambda) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(\lambda-\mu)K_1^-(\lambda)R_{21}(\lambda+\mu)K_2^-(\mu) = K_2^-(\mu)R_{12}(\lambda+\mu)K_1^-(\lambda)R_{21}(\lambda-\mu).$$

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Dual reflection equation

Due to the properties of the Yang R-matrix the dual reflection equation can be presented in the following form

$$R_{12}(-\lambda+\mu)K_1^+(\lambda)R_{21}(-\lambda-\mu-2\eta)K_2^+(\mu) =$$
$$K_2^+(\mu)R_{12}(-\lambda-\mu-2\eta)K_1^+(\lambda)R_{21}(-\lambda+\mu).$$

One can then verify that the mapping

$$K^+(\lambda) = K^-(-\lambda-\eta)$$

is a bijection between solutions of the reflection equation and the dual reflection equation.

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$K^{\mp}(\lambda)$ matrices

The general, spectral parameter dependent solutions of the reflection equation and the dual reflection equation can be written as follows

$$\widetilde{K}^{-}(\lambda) = \begin{pmatrix} \xi^{-} - \lambda & \widetilde{\psi}^{-}\lambda \\ \widetilde{\phi}^{-}\lambda & \xi^{-} + \lambda \end{pmatrix},$$

$$\widetilde{K}^{+}(\lambda) = \begin{pmatrix} \xi^{+} + \lambda + \eta & -\widetilde{\psi}^{+}(\lambda + \eta) \\ -\widetilde{\phi}^{+}(\lambda + \eta) & \xi^{+} - \lambda - \eta \end{pmatrix}.$$

$K^{\mp}(\lambda)$ matrices

We notice the condition

$$\left(\widetilde{\phi}^{-}\widetilde{\psi}^{+}-\widetilde{\phi}^{+}\widetilde{\psi}^{-}\right)^{2}=4\left(\widetilde{\phi}^{-}-\widetilde{\phi}^{+}\right)\left(\widetilde{\psi}^{-}-\widetilde{\psi}^{+}\right)$$

has to be imposed on the parameters of the K-matrices so that they are upper triangularizable by a single similarity matrix M. When the square root with the negative sign is taken on the right-hand-side of the equation above then one possible choice for M is given by

$$M=\begin{pmatrix} -1-\nu^{-} & \widetilde{\phi}^{-} \\ \widetilde{\phi}^{-} & -1-\nu^{-} \end{pmatrix},$$

where $\nu^{-}=\sqrt{1+\widetilde{\phi}^{-}\widetilde{\psi}^{-}}$.

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where $\nu^{-}=\sqrt{1+\widetilde{\phi}^{-}\widetilde{\psi}^{-}}$.

$K^{\mp}(\lambda)$ matrices

Evidently this matrix does not depend on the spectral parameter λ and it is such that

$$K^{-}(\lambda) = M^{-1}\widetilde{K}^{-}(\lambda)M = \begin{pmatrix} \xi^{-} - \lambda\nu^{-} & \lambda\psi^{-} \\ 0 & \xi^{-} + \lambda\nu^{-} \end{pmatrix},$$

$$K^{+}(\lambda) = M^{-1}\widetilde{K}^{+}(\lambda)M = \begin{pmatrix} \xi^{+} + (\lambda+\eta)\nu^{+} & -\psi^{+}(\lambda+\eta) \\ 0 & \xi^{+} - (\lambda+\eta)\nu^{+} \end{pmatrix},$$

with $\psi^{-} = \widetilde{\phi}^{-} + \widetilde{\psi}^{-}$, $\nu^{+} = \sqrt{1 + \widetilde{\phi}^{+}\widetilde{\psi}^{+}}$ and $\psi^{+} = \widetilde{\phi}^{+} + \widetilde{\psi}^{+}$.

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Monodromy matrix $\mathcal{T}(\lambda)$

We use the Sklyanin approach to integrable spin chains with non-periodic boundary conditions. The monodromy matrix $\mathcal{T}(\lambda)$ is

$$\mathcal{T}_0(\lambda) = T_0(\lambda) K_0^-(\lambda) \widetilde{T}_0(\lambda),$$

it consists of the matrix $T(\lambda)$, a reflection matrix $K^-(\lambda)$ and the matrix

$$\widetilde{T}_0(\lambda) = \left(\begin{array}{cc} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{array} \right) = \mathbb{L}_{01}(\lambda + \alpha_1 + \eta) \cdots \mathbb{L}_{0N}(\lambda + \alpha_N + \eta).$$

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Monodromy matrix $\mathcal{T}(\lambda)$

The RTT-relations for the matrix $\widetilde{T}_0(\lambda)$ can be recast as follows

$$\widetilde{T}_{0'}(\mu) R_{00'}(\lambda+\mu) T_0(\lambda) = T_0(\lambda) R_{00'}(\lambda+\mu) \widetilde{T}_{0'}(\mu),$$
$$\widetilde{T}_0(\lambda) \widetilde{T}_{0'}(\mu) R_{00'}(\mu-\lambda) = R_{00'}(\mu-\lambda) \widetilde{T}_{0'}(\mu) \widetilde{T}_0(\lambda).$$

Then, by construction, the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$ are

$$R_{00'}(\lambda-\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda+\mu)\mathcal{T}_{0'}(\mu) = \mathcal{T}_{0'}(\mu)R_{00'}(\lambda+\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda-\mu).$$

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Monodromy matrix $\mathcal{T}(\lambda)$

From the equation above we can read off the commutation relations of the entries of the monodromy matrix

$$\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}.$$

Following Sklyanin we introduce the operator

$$\widehat{\mathcal{D}}(\lambda) = \mathcal{D}(\lambda) - \frac{\eta}{2\lambda + \eta}\mathcal{A}(\lambda).$$

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Sklyanin determinant

The exchange relations admit a central element, the so-called Sklyanin determinant,

$$\Delta\left[\mathcal{T}(\lambda)\right] = \mathrm{tr}_{00'} P^{-}_{00'} \mathcal{T}_0(\lambda - \eta/2) R_{00'}(2\lambda) \mathcal{T}_{0'}(\lambda + \eta/2).$$

The element $\Delta\left[\mathcal{T}(\lambda)\right]$ can be expressed in form

$$\Delta\left[\mathcal{T}(\lambda)\right] = 2\lambda \widehat{\mathcal{D}}(\lambda - \eta/2)\mathcal{A}(\lambda + \eta/2) - (2\lambda + \eta)\mathcal{B}(\lambda - \eta/2)\mathcal{C}(\lambda + \eta/2).$$

Transfer matrix

The open chain transfer matrix is given by the trace of the monodromy $\mathcal{T}(\lambda)$ over the auxiliary space V_0 with an extra reflection matrix $K^+(\lambda)$,

$$t(\lambda) = \mathrm{tr}_0 \left(K^+(\lambda) \mathcal{T}(\lambda) \right) .$$

The reflection matrix $K^+(\lambda)$ is the corresponding solution of the dual reflection equation.

The commutativity of the transfer matrix for different values of the spectral parameter

$$[t(\lambda), t(\mu)] = 0,$$

is guaranteed by the dual reflection equation and the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$.

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Algebraic Bethe Ansatz

The main aim of this section is to define the Bethe vectors as to obtain the most simplest formulae for the off shell action of the transfer matrix of the spin chain on these Bethe vectors.

The first step in this direction is to get the expressions of the entries of the monodromy matrix $\mathcal{T}(\lambda)$ in terms of the corresponding ones of the monodromies $T(\lambda)$ and $\widetilde{T}(\lambda)$.

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Monodromy matrix $\mathcal{T}(\lambda)$

According to definition of the monodromy matrix $\mathcal{T}(\lambda)$ we have

$$\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} \xi^- - \lambda\nu^- & \psi^-\lambda \\ 0 & \xi^- + \lambda\nu^- \end{pmatrix} \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix}.$$

Entries of the monodromy matrix $\mathcal{T}(\lambda)$

From the equation above, and also using the RTT-relations, we obtain

$$\mathcal{A}(\lambda) = (\xi^- - \lambda\nu^-)A(\lambda)\widetilde{A}(\lambda) + \left((\psi^-\lambda)A(\lambda) + (\xi^- + \lambda\nu^-)B(\lambda)\right)\widetilde{C}(\lambda)$$

$$\mathcal{D}(\lambda) = (\xi^- - \lambda\nu^-)\left(\widetilde{B}(\lambda)C(\lambda) - \frac{\eta}{2\lambda+\eta}\left(D(\lambda)\widetilde{D}(\lambda) - \widetilde{A}(\lambda)A(\lambda)\right)\right) + \left((\psi^-\lambda)C(\lambda) + (\xi^- + \lambda\nu^-)D(\lambda)\right)\widetilde{D}(\lambda)$$

$$\mathcal{B}(\lambda) = (\xi^- - \lambda\nu^-)\left(\frac{2\lambda}{2\lambda+\eta}\widetilde{B}(\lambda)A(\lambda) - \frac{\eta}{2\lambda+\eta}B(\lambda)\widetilde{D}(\lambda)\right) + \left((\psi^-\lambda)A(\lambda) + (\xi^- + \lambda\nu^-)B(\lambda)\right)\widetilde{D}(\lambda)$$

$$\mathcal{C}(\lambda) = (\xi^- - \lambda\nu^-)C(\lambda)\widetilde{A}(\lambda) + \left((\psi^-\lambda)C(\lambda) + (\xi^- + \lambda\nu^-)D(\lambda)\right)\widetilde{C}(\lambda).$$

Entries of the monodromy matrix $\mathcal{T}(\lambda)$

With the aim of obtaining the action of the operators $\mathcal{A}(\lambda)$, $\mathcal{D}(\lambda)$ and $\mathcal{C}(\lambda)$ on the vector Ω_+ we first observe that the action of the operators $\widetilde{A}(\lambda)$, $\widetilde{D}(\lambda)$ and $\widetilde{C}(\lambda)$ on the vector Ω_+ is

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\begin{aligned}
\widetilde{A}(\lambda)\Omega_+ &= \widetilde{a}(\lambda)\Omega_+, \quad \text{with} \quad \widetilde{a}(\lambda) = \prod_{m=1}^{N} \frac{\lambda + \alpha_m + \eta + \eta s_m}{\lambda + \alpha_m + \eta}, \\
\widetilde{D}(\lambda)\Omega_+ &= \widetilde{d}(\lambda)\Omega_+, \quad \text{with} \quad \widetilde{d}(\lambda) = \prod_{m=1}^{N} \frac{\lambda + \alpha_m + \eta - \eta s_m}{\lambda + \alpha_m + \eta}, \\
\widetilde{C}(\lambda)\Omega_+ &= 0.
\end{aligned}
$$

Operators $\mathcal{A}(\lambda)$, $\mathcal{D}(\lambda)$ and $\mathcal{C}(\lambda)$ act on Ω_+

Finally, we derive

$$\begin{aligned}
\mathcal{C}(\lambda)\Omega_+ &= 0, \\
\mathcal{A}(\lambda)\Omega_+ &= \alpha(\lambda)\Omega_+, \quad \text{with} \quad \alpha(\lambda) = (\xi^- - \lambda\nu^-)a(\lambda)\widetilde{a}(\lambda), \\
\mathcal{D}(\lambda)\Omega_+ &= \delta(\lambda)\Omega_+, \quad \text{with}
\end{aligned}$$

$$\delta(\lambda) = \left((\xi^- + \lambda\nu^-) - \frac{\eta}{2\lambda+\eta}(\xi^- - \lambda\nu^-)\right) d(\lambda)\widetilde{d}(\lambda) + \frac{\eta}{2\lambda+\eta}(\xi^- - \lambda\nu^-)a(\lambda)\widetilde{a}(\lambda).$$

Operator $\widehat{\mathcal{D}}(\lambda)$ acts on Ω_+

In what follows we will use the fact that Ω_+ is an eigenvector of the operator $\widehat{\mathcal{D}}(\lambda)$

$$\widehat{\mathcal{D}}(\lambda)\Omega_+ = \widehat{\delta}(\lambda)\Omega_+, \quad \text{with} \quad \widehat{\delta}(\lambda) = \delta(\lambda) - \frac{\eta}{2\lambda+\eta}\alpha(\lambda),$$

or explicitly

$$\widehat{\delta}(\lambda) = \left((\xi^- + \lambda\nu^-) - \frac{\eta}{2\lambda+\eta}(\xi^- - \lambda\nu^-)\right) d(\lambda)\widetilde{d}(\lambda).$$

Transfer matrix

The transfer matrix of the inhomogeneous XXX chain

$$t(\lambda) = \mathrm{tr}\, K^{+}(\lambda)\mathcal{T}(\lambda),$$

with the triangular K^{+} matrix, can be expressed using $\widehat{\mathcal{D}}(\lambda)$ operator

$$t(\lambda) = \kappa_1(\lambda)\mathcal{A}(\lambda) + \kappa_2(\lambda)\widehat{\mathcal{D}}(\lambda) + \kappa_{12}(\lambda)\mathcal{C}(\lambda),$$

with

$$\kappa_1(\lambda) = 2(\xi^{+}+\lambda\nu^{+})\,\frac{\lambda+\eta}{2\lambda+\eta},\ \kappa_2(\lambda) = \xi^{+}-(\lambda+\eta)\nu^{+},\ \kappa_{12}(\lambda) = -\psi^{+}(\lambda+\eta).$$

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Ω_+ is an eigenvector of $t(\lambda)$

Evidently the vector Ω_+ is an eigenvector of the transfer matrix

$$t(\lambda)\Omega_+ = \left(\kappa_1(\lambda)\alpha(\lambda) + \kappa_2(\lambda)\widehat{\delta}(\lambda)\right)\Omega_+ = \Lambda_0(\lambda)\Omega_+ .$$

For simplicity we have suppressed the dependence of the eigenvalue $\Lambda_0(\lambda)$ on the boundary parameters ξ^+ and ν^+ as well as the quasi-classical parameter η.

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Bethe vector Ψ_1

We proceed to define the Bethe vectors $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$ as to make the off shell action of $t(\lambda)$ on them as simple as possible.

Our next step is to show that

$$\Psi_1(\mu) = \mathcal{B}(\mu)\Omega_+ + b_1(\mu)\Omega_+,$$

is a Bethe vector,if $b_1(\mu)$ is chosen to be

$$b_1(\mu) = \frac{\psi^+}{2\nu^+}\left(\frac{2\mu}{2\mu+\eta}\alpha(\mu) - \widehat{\delta}(\mu)\right).$$

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Bethe vector Ψ_1

A straightforward calculation shows that the off shell action of the transfer matrix on $\Psi_1(\mu)$ is given by

$$t(\lambda)\Psi_1(\mu) = \Lambda_1(\lambda,\mu)\Psi_1(\mu) + \frac{2\eta(\lambda+\eta)(\xi^+ + \mu\nu^+)}{(\lambda-\mu)(\lambda+\mu+\eta)} F_1(\mu)\Psi_1(\lambda)$$

where the eigenvalue $\Lambda_1(\lambda,\mu)$ is given by

$$\Lambda_1(\lambda,\mu) = \kappa_1(\lambda)\frac{(\lambda+\mu)(\lambda-\mu-\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)}\alpha(\lambda)+\kappa_2(\lambda)\frac{(\lambda-\mu+\eta)(\lambda+\mu+2\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)}\widehat{\delta}(\lambda).$$

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$$\Lambda_1(\lambda,\mu) = \kappa_1(\lambda)\frac{(\lambda+\mu)(\lambda-\mu-\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)}\alpha(\lambda) + \kappa_2(\lambda)\frac{(\lambda-\mu+\eta)(\lambda+\mu+2\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)}\widehat{\delta}(\lambda).$$

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Bethe vector Ψ_1

The unwanted term of the off shell action is annihilated by the Bethe equation

$$F_1(\mu) = \frac{2\mu}{2\mu + \eta}\alpha(\mu) - \frac{\xi^+ - (\mu + \eta)\nu^+}{\xi^+ + \mu\nu^+}\widehat{\delta}(\mu) = 0,$$

or equivalently,

$$\frac{\alpha(\mu)}{\widehat{\delta}(\mu)} = \frac{(\mu + \eta)\kappa_2(\mu)}{\mu\,\kappa_1(\mu)} = \frac{(2\mu + \eta)(\xi^+ - (\mu + \eta)\nu^+)}{2\mu\,(\xi^+ + \mu\nu^+)}.$$

Bethe vector Ψ_2

We seek $\Psi_2(\mu_1, \mu_2)$ in the form

$$\Psi_2(\mu_1,\mu_2) = \mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ + b_2^{(1)}(\mu_2;\mu_1)\mathcal{B}(\mu_1)\Omega_+ + b_2^{(1)}(\mu_1;\mu_2)\mathcal{B}(\mu_2)\Omega_+ + b_2^{(2)}(\mu_1,\mu_2)\Omega_+,$$

where $b_2^{(1)}(\mu_1;\mu_2)$ and $b_2^{(2)}(\mu_1,\mu_2)$ are given by

$$b_2^{(1)}(\mu_1;\mu_2) = \frac{\psi^+}{2\nu^+}\left(\frac{2\mu_1}{2\mu_1+\eta}\frac{(\mu_1+\mu_2)(\mu_1-\mu_2-\eta)}{(\mu_1-\mu_2)(\mu_1+\mu_2+\eta)}\alpha(\mu_1) - \frac{(\mu_1-\mu_2+\eta)(\mu_1+\mu_2+2\eta)}{(\mu_1-\mu_2)(\mu_1+\mu_2+\eta)}\widehat{\delta}(\mu_1)\right),$$

$$b_2^{(2)}(\mu_1,\mu_2) = \frac{1}{2}\left(b_2^{(1)}(\mu_1;\mu_2)\, b_1(\mu_2) + b_2^{(1)}(\mu_2;\mu_1)\, b_1(\mu_1)\right).$$

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where $b_2^{(1)}(\mu_1; \mu_2)$ and $b_2^{(2)}(\mu_1, \mu_2)$ are given by

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$$b_2^{(2)}(\mu_1, \mu_2) = \frac{1}{2}\left(b_2^{(1)}(\mu_1; \mu_2)\, b_1(\mu_2) + b_2^{(1)}(\mu_2; \mu_1)\, b_1(\mu_1)\right).$$

Bethe vector Ψ_2

It can be shown that the action of $t(\lambda)$ on $\Psi_2(\mu_1, \mu_2)$ reads

$$t(\lambda)\Psi_2(\mu_1, \mu_2) = \Lambda_2(\lambda, \{\mu_i\})\Psi_2(\mu_1, \mu_2)$$
$$+ \sum_{i=1}^{2} \frac{2\eta(\lambda+\eta)(\xi^+ + \mu_i\nu^+)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)} F_2(\mu_i; \mu_{3-i})\Psi_2(\lambda, \mu_{3-i}),$$

Bethe vector Ψ_2

where the eigenvalue is given by

$$\Lambda_2(\lambda, \{\mu_i\}) = \kappa_1(\lambda)\,\alpha(\lambda)\,\prod_{i=1}^{2} \frac{(\lambda+\mu_i)(\lambda-\mu_i-\eta)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)} + \kappa_2(\lambda)\,\widehat{\delta}(\lambda)\,\prod_{i=1}^{2} \frac{(\lambda-\mu_i+\eta)(\lambda+\mu_i+2\eta)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)}$$

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Bethe vector Ψ_2

and the two unwanted terms are canceled by the Bethe equations which follow from $F_2(\mu_i;\mu_{3-i}) = 0$, i. e.

$$\frac{\alpha(\mu_i)}{\widehat{\delta}(\mu_i)} = \frac{(\mu_i+\eta)\kappa_2(\mu_i)}{\mu_i\,\kappa_1(\mu_i)}\frac{(\mu_i-\mu_{3-i}+\eta)(\mu_i+\mu_{3-i}+2\eta)}{(\mu_i+\mu_{3-i})(\mu_i-\mu_{3-i}-\eta)},$$

$i = \{1, 2\}$.

Outline

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Gaudin model

We explore further the results obtained in the previous section on the XXX Heisenberg spin chain in the case when both boundary matrix are upper-triangular.

We combine them together with the quasi-classical limit with the aim of implementing fully the off shell Bethe ansatz for the corresponding Gaudin model by defining the Bethe vectors and deriving its spectrum and the corresponding Bethe equations.

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Gaudin K-matrix

For the study of the open Gaudin model we impose

$$\lim_{\eta \to 0} \left(K^{+}(\lambda) K^{-}(\lambda) \right) = \left(\xi^2 - \lambda^2 \nu^2 \right) \mathbb{1}.$$

In particular, this implies that the parameters of the reflection matrices on the left and on the right end of the chain are the same.

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Gaudin K-matrix

Therefore, we will write

$$K^{-}(\lambda) \equiv K(\lambda) = \begin{pmatrix} \xi - \lambda\nu & \lambda\psi \\ 0 & \xi + \lambda\nu \end{pmatrix},$$

so that

$$K^{+}(\lambda) = K(-\lambda - \eta) = \begin{pmatrix} \xi + (\lambda + \eta)\nu & -\psi(\lambda + \eta) \\ 0 & \xi - (\lambda + \eta)\nu \end{pmatrix}.$$

Quasi-classical expansion

The quasi-classical expansion reads

$$2\lambda t(\lambda) - \Delta\left[\mathcal{T}(\lambda)\right] = 2\lambda\left(\xi^2 - \lambda^2\nu^2\right)\mathbb{1} + \eta\left(\xi^2 - 3\lambda^2\nu^2\right)\mathbb{1}$$
$$+ \eta^2\lambda\left(\left(\xi^2 - \lambda^2\nu^2\right)\tau(\lambda) - \frac{\nu^2}{2}\mathbb{1}\right) + \mathcal{O}(\eta^3),$$

where $\tau(\lambda)$ is the generating function for the Gaudin model with upper triangular boundary matrix

$$\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda),$$

Quasi-classical expansion

with the Lax matrix

$$\mathcal{L}_0(\lambda) = \sum_{m=1}^{N} \left(\frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{\vec{\sigma}_0 \cdot \left(K_m^{-1}(\lambda) \vec{S}_m K_m(\lambda) \right)}{\lambda + \alpha_m} \right).$$

The Gaudin Hamiltonians with the boundary terms are obtained from the residues of the generating function at poles $\lambda = \pm\alpha_m$:

$$\mathrm{Res}_{\lambda=\alpha_m} \tau(\lambda) \ = \ 4\, H_m \quad \text{and} \quad \mathrm{Res}_{\lambda=-\alpha_m} \tau(\lambda) \ = \ 4\, \widetilde{H}_m.$$

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Sklyanin determinant

The element $\Delta\left[\mathcal{T}(\lambda)\right]$ can be written in form

$$\Delta\left[\mathcal{T}(\lambda)\right] = 2\lambda\widehat{\mathcal{D}}(\lambda-\eta/2)\mathcal{A}(\lambda+\eta/2) - (2\lambda+\eta)\mathcal{B}(\lambda-\eta/2)\mathcal{C}(\lambda+\eta/2).$$

Therefore Ω_+ is an eigenvector

$$\Delta\left[\mathcal{T}(\lambda)\right]\Omega_+ = \alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\Omega_+.$$

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Ω_+ is an eigenvector

Moreover, Ω_+ is an eigenvector

$$\left(2\lambda t(\lambda) - \Delta\left[\mathcal{T}(\lambda)\right]\right)\Omega_+ = 2\lambda\left(\Lambda_0(\lambda) - \alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\right)\Omega_+ .$$

We can expand

$$2\lambda\left(\kappa_1(\lambda)\alpha(\lambda) + \kappa_2(\lambda)\widehat{\delta}(\lambda) - \alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\right) = 2\lambda\left(\xi^2 - \lambda^2\nu^2\right)$$

$$+\,\eta\left(\xi^2 - 3\lambda^2\nu^2\right) + \eta^2\lambda\left(\left(\xi^2-\lambda^2\nu^2\right)\chi_0(\lambda) - \frac{\nu^2}{2}\right) + \mathcal{O}(\eta^3).$$

Ω_+ is an eigenvector

Moreover, Ω_+ is an eigenvector

$$(2\lambda t(\lambda) - \Delta\left[\mathcal{T}(\lambda)\right])\,\Omega_+ = 2\lambda\left(\Lambda_0(\lambda) - \alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\right)\Omega_+ .$$

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Ω_+ is an eigenvector

It follows that Ω_+ is an eigenvector of the generating function

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+,$$

with

$$\chi_0(\lambda) = \frac{4\lambda}{\xi^2 - \lambda^2\nu^2}\sum_{m=1}^{N}\left(\frac{s_m}{\lambda-\alpha_m} + \frac{s_m}{\lambda+\alpha_m}\right) + 2\times$$
$$\times\sum_{m,n=1}^{N}\left(\frac{s_m s_n + s_m\delta_{mn}}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{2\left(s_m s_n + s_m\delta_{mn}\right)}{(\lambda-\alpha_m)(\lambda+\alpha_n)} + \frac{s_m s_n + s_m\delta_{mn}}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right).$$

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General $\chi_M(\lambda, \mu_1, \ldots, \mu_M)$

In general, we can obtain the spectrum $\chi_M(\lambda, \mu_1, \ldots, \mu_M)$ of the generating function $\tau(\lambda)$ of the Gaudin Hamiltonians through the expansion

$$2\lambda \left(\Lambda_M(\lambda, \mu_1, \ldots, \mu_M) - \alpha(\lambda + \eta/2)\, \widehat{\delta}(\lambda - \eta/2) \right) = 2\lambda \left(\xi^2 - \lambda^2\nu^2\right)$$

$$+ \eta \left(\xi^2 - 3\lambda^2\nu^2\right) + \eta^2 \lambda \left(\left(\xi^2 - \lambda^2\nu^2\right) \chi_M(\lambda, \mu_1, \ldots, \mu_M) - \frac{\nu^2}{2} \right) + \mathcal{O}(\eta^3),$$

or explicitly

General $\chi_M(\lambda, \mu_1, \ldots, \mu_M)$

$$
\begin{aligned}
\chi_M(\lambda, \mu_1, \ldots, \mu_M) &= \frac{-4\lambda^2\nu^4}{(\xi^2 - \lambda^2\nu^2)^2} \\
&+ 2\sum_{j,k=1}^{M} \left(\frac{1-\delta_{jk}}{(\lambda-\mu_j)(\lambda-\mu_k)} + \frac{2\,(1-\delta_{jk})}{(\lambda-\mu_j)(\lambda+\mu_k)} + \frac{1-\delta_{jk}}{(\lambda+\mu_j)(\lambda+\mu_k)} \right) \\
&+ 2\sum_{m,n=1}^{N} \left(\frac{s_m s_n + s_m\delta_{mn}}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{2\,(s_m s_n + s_m\delta_{mn})}{(\lambda-\alpha_m)(\lambda+\alpha_n)} + \frac{s_m s_n + s_m\delta_{mn}}{(\lambda+\alpha_m)(\lambda+\alpha_n)} \right) \\
&- 4\left(\sum_{j=1}^{M} \left(\frac{1}{\lambda-\mu_j} + \frac{1}{\lambda+\mu_j} \right) - \frac{\lambda\nu^2}{\xi^2-\lambda^2\nu^2} \right) \times \\
&\times \left(\sum_{m=1}^{N} \left(\frac{s_m}{\lambda-\alpha_m} + \frac{s_m}{\lambda+\alpha_m} \right) + \frac{\lambda\nu^2}{\xi^2-\lambda^2\nu^2} \right).
\end{aligned}
$$

Expansion of $F_M(\mu_1; \mu_2, \dots, \mu_M)$

As our next important step toward obtaining the formulas of the algebraic Bethe ansatz for the corresponding Gaudin model we observe that the first term in the expansion of the function $F_M(\mu_1; \mu_2, \dots, \mu_M)$ in powers of η is

$$F_M(\mu_1; \mu_2, \dots, \mu_M) = \eta f_M(\mu_1; \mu_2, \dots, \mu_M) + \mathcal{O}(\eta^2),$$

where

Expansion of $F_M(\mu_1; \mu_2, \ldots, \mu_M)$

$$f_M(\mu_1; \mu_2, \ldots, \mu_M) = \frac{2\mu_1\nu^2}{\xi + \mu_1\nu} - 2(\xi - \mu_1\nu) \sum_{j=2}^{M} \left(\frac{1}{\mu_1 - \mu_j} + \frac{1}{\mu_1 + \mu_j} \right)$$
$$+ 2(\xi - \mu_1\nu) \sum_{m=1}^{N} \left(\frac{s_m}{\mu_1 - \alpha_m} + \frac{s_m}{\mu_1 + \alpha_m} \right).$$

Bethe vector $\varphi_1(\mu)$ of the Gaudin model

We have used the expand the Bethe vector $\Psi_1(\mu)$ of the Heisenberg spin chain in powers of η and obtained the Bethe vector $\varphi_1(\mu)$ of the Gaudin model

$$\Psi_1(\mu) = \eta\varphi_1(\mu) + \mathcal{O}(\eta^2),$$

where

$$\varphi_1(\mu) = \sum_{m=1}^{N} \left(\frac{\xi + \alpha_m \nu}{\mu - \alpha_m} + \frac{\xi + \alpha_m \nu}{\mu + \alpha_m} \right) \left(\frac{\psi\, S_m}{\nu} + S_m^- \right) \Omega_+ .$$

Off shell for the Gaudin model

As our final step we observe that we have the off shell action of the difference of the transfer matrix of the XXX chain and the central element, the so-called Sklyanin determinant, on the Bethe vector $\Psi_1(\mu)$

$$
\begin{aligned}
(2\lambda t(\lambda) - \Delta\left[\mathcal{T}(\lambda)\right])\,\Psi_1(\mu) &= 2\lambda\left(\Lambda_1(\lambda,\mu) - \alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\right)\Psi_1(\mu)\\
&+ (2\lambda)\frac{2\eta(\lambda+\eta)(\xi+\mu\nu)}{(\lambda-\mu)(\lambda+\mu+\eta)}F_1(\mu)\Psi_1(\lambda).
\end{aligned}
$$

Off shell for the Gaudin model

Finally, the off shell action of the generating function the Gaudin Hamiltonians on the vector $\varphi_1(\mu)$ can be obtained from the equation above by using the expansions above

$$\tau(\lambda)\varphi_1(\mu) = \chi_1(\lambda,\mu)\varphi_1(\mu) + \frac{4\lambda(\xi+\mu\nu)}{(\xi^2-\lambda^2\nu^2)(\lambda^2-\mu^2)} f_1(\mu)\varphi_1(\lambda).$$

Therefore $\varphi_1(\mu)$ is the Bethe vector of the corresponding Gaudin model, i.e. the eigenvector of the generating function the Gaudin Hamiltonians once the unwanted term is canceled by imposing the corresponding Bethe equation

$$f_1(\mu) = \frac{2\mu\nu^2}{\xi+\mu\nu} + 2(\xi-\mu\nu)\sum_{m=1}^{N}\left(\frac{s_m}{\mu-\alpha_m}+\frac{s_m}{\mu+\alpha_m}\right) = 0.$$

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Outline

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Summary

- We have mplemented fully the algebraic Bethe ansatz for the XXX Heisenberg spin chain in the case when both boundary matrices can be brought to the upper-triangular form.
- We have defined the Bethe vectors which yield the strikingly simple expression for the off shell action of the transfer matrix, deriving the spectrum and the corresponding Bethe equations.
- We have explored further these results by obtaining the off shell action of the generating function of the Gaudin Hamiltonians on the Bethe vectors through the so-called quasi-classical limit.

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- To compare the results obtained via quasi-classical limit and the ones coming from the algebraic Bethe ansatz for the Gaudin model.
- The study of the XXZ case...

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