# Vertex Operator Approach to Semi-Infinite Lattice Model

Takeo Kojima

Yamagata University, Japan

8th Mathematical Physics Meeting

24-31 August 2014

Belgrade, Serbia

$\boxed{\text{Outline}}$

Correlation function of Semi-Infinite Lattice

(1) Symmetry

Boundary $sl_N$-ABF model ( 2d. Ising / XXZ chain )

$U_q(\widehat{sl_2}) \longmapsto U_{qp}(\widehat{sl_N})$ elliptic algebra

(2) Boundary condition

Boundary XXZ chain $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$h\sigma^z \longmapsto h\sigma^z + k\sigma^+$ Triangle $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

## Symmetry

(1) Boundary $sl_N$ - ABF model

- $U_{qp}(\widehat{sl_N})$ elliptic algebra
- Correlation function

$$\boxed{U_{qp}(\widehat{sl_N})}$$

$\boxed{\text{Lie algebra } \mathfrak{sl}_2}$

- $\mathfrak{sl}_2 = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \mathrm{tr}(X) = a + d = 0 \right\}$
- $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\boxed{\begin{aligned} &[h, e] = 2e, \quad [h, f] = -2f \\ &[e, f] = h \end{aligned}}$$

$$[X, Y] = XY - YX$$

$\boxed{\text{Quantum group } U_q(sl_2)}$ $\quad q \neq 0, \pm 1$

$\boxed{\text{Generators}}$ $\quad e, f, q^h$

$\boxed{\text{Relations}}$

$$q^h e q^{-h} = q^2 e$$

$$q^h f q^{-h} = q^{-2} f$$

$$[e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}$$

Quantum group $U_q(\widehat{sl_2})$ [Jimbo] (1985)

Generators $e_0, e_1, f_0, f_1, q^{h_0}, q^{h_1}$ $\quad [n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

Relations

- $q^{h_i} e_j q^{-h_i} = q^{A_{ij}} e_j$ $\quad (A_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$
- $q^{h_i} f_j q^{-h_i} = q^{-A_{ij}} f_j$
- $[e_i, f_j] = \delta_{ij} \dfrac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$
- $e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i + e_j e_i^3 = 0$

$(e_i \leftrightarrow f_i)$ $\quad (i, j = 0, 1)$

## Quantum group $U_q(\widehat{\mathfrak{sl}_2})$

[Drinfeld] (1988)

**Generators**

$a_m,\ x_n^{\pm},\ h_1,\ c \quad (m, n \in \mathbb{Z},\ m \neq 0)$

$$X^{\pm}(z) = \sum_{n \in \mathbb{Z}} x_n^{\pm} z^{-n-1}$$

**Relations**

- $$[a_m, X^{\pm}(z)] = \pm \frac{[2m]_q}{m} q^{\mp c \frac{|m|}{2}} z^m X^{\pm}(z)$$
- $$(z_1 - q^{\pm 2} z_2) X^{\pm}(z_1) X^{\pm}(z_2) = (q^{\pm 2} z_1 - z_2) X^{\pm}(z_2) X^{\pm}(z_1)$$
- $$[X^+(z_1), X^-(z_2)]$$

$$= \frac{1}{(q - q^{-1}) z_1 z_2} \Big( \delta\Big(q^{-c} \frac{z_1}{z_2}\Big) q^{h_1} \exp\Big((q - q^{-1}) \sum_{m=1}^{\infty} a_m q^{\frac{cm}{2}} z^{-m}\Big)$$

$$- \delta\Big(q^{c} \frac{z_1}{z_2}\Big) q^{-h_1} \exp\Big(-(q - q^{-1}) \sum_{m=1}^{\infty} a_{-m} q^{\frac{cm}{2}} z^{m}\Big) \Big)$$

## Two realizations of $U_q(\widehat{\mathfrak{sl}_2})$

### Correspondence

$$x_0^+ = e_1 , \quad x_0^- = f_1$$
$$x_{-1}^+ = q^{-h_1} f_0 , \quad x_{+1}^- = e_0 q^{h_1}$$

Quantum group | polynomial

$$(z_1 - q^2 z_2)\, X^+(z_1)\, X^+(z_2) = (q^2 z_1 - z_2)\, X^+(z_2)\, X^+(z_1)$$

$\Downarrow$ Deformation

Elliptic algebra | theta function

$$[u_1 - u_2 + 1]\, F(u_1)\, F(u_2) = [u_1 - u_2 - 1]\, F(u_2)\, F(u_1)$$

$$[u] = q^{\frac{u^2}{r} - u}\, \Theta_{q^{2r}}(q^{2u})$$

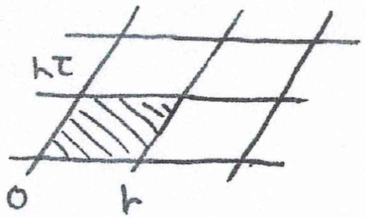
$$\Theta_p(z) = (p;p)_\infty (z;p)_\infty (p/z;p)_\infty\,, \quad (z;p)_\infty = \prod_{m=0}^{\infty} (1 - p^m z)$$

## Theta function

$(z = q^{2u},\ \tau = \pi i/\log q)$

- $[u] = q^{\frac{u^2}{r} - u}\, \Theta_{q^{2r}}(q^{2u})$
- $\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty$

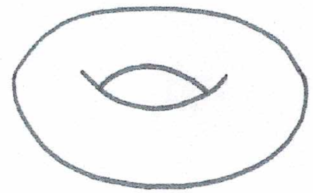
$$(z;p)_\infty = \prod_{m=0}^{\infty}(1 - p^m z).$$




## Quasi-Periodicity

holomorphic

- $[u+r] = -[u]$
- $[u+r\tau] = -e^{2\pi i(u+\frac{\tau}{2})/r}\,[u]$



Torus

## Elliptic algebra $U_{q,p}(\widehat{sl_N})$

[Kojima, Konno] (2003)

$$0<q<1\,,\quad 0<p=q^{2r}<1 \quad (r>0)\,,\quad z=q^{2u}$$

**Relations**

$$E_j(z)\,,\ F_j(z)\,,\ H_j(z) \qquad (1\le j\le N-1)$$

- $\left[u_1-u_2-\frac{s}{N}\right] F_j(z_1)F_{j+1}(z_2)=\left[u_2-u_1+\frac{s}{N}-1\right]F_{j+1}(z_2)F_j(z_1)$
- $\left[u_1-u_2+1\right] F_j(z_1)F_j(z_2)=\left[u_2-u_1+1\right]F_j(z_2)F_j(z_1)$
- $F_i(z_1)F_j(z_2)=F_j(z_2)F_i(z_1)$ $\quad (i,j)$ otherwise

$$[u]=q^{\frac{u^2}{r}-u}\,\Theta_{q^{2r}}(q^{2u})$$

## Relations

- $[u_1-u_2+1-\frac{s}{N}]^* E_{\bar{j}}(z_1)E_{\overline{j+1}}(z_2) = [u_2-u_1+\frac{s}{N}]^* E_{\overline{j+1}}(z_2)E_{\bar{j}}(z_1)$
- $[u_1-u_2-1]^* E_{\bar{j}}(z_1)E_{\bar{j}}(z_2) = [u_1-u_2+1]^* E_{\bar{j}}(z_2)E_{\bar{j}}(z_1)$

    - $E_{\bar{i}}(z_1)E_{\bar{j}}(z_2) = E_{\bar{j}}(z_2)E_{\bar{i}}(z_1)$

- $[E_{\bar{i}}(z_1), F_{\bar{j}}(z_2)] = \frac{\delta_{\bar{i}\bar{j}}}{q-q^{-1}} \times$

$\times \left( \delta(q^c \frac{z_2}{z_1}) H_{\bar{i}}(q^r z_2) - \delta(q^c \frac{z_1}{z_2}) H_{\bar{i}}(q^{-r} z_2) \right)$ etc.

- $[u]^* = q^{\frac{u^2}{r^*}-u} H_{q^{2r^*}}(z)$ $\qquad r^* = r-c > 0$
- $[u] = q^{\frac{u^2}{r}-u} H_{q^{2r}}(z)$ $\qquad r^* \neq r$

## Bosonization

$U_{q,p}(\widehat{sl_N})$ Level $c=1$

### Boson

$$B_m^i \quad (i=1,2,\cdots,N) \quad (m\in\mathbb{Z}_{\neq 0})$$

$$[B_m^i, B_n^j] = m\frac{[(r-1)m]_q}{[rm]_q} \times \begin{cases} \dfrac{[(s-1)m]_q}{[sm]_q} & (i=j) \\ (-1)\dfrac{[m]_q}{[sm]_q} q^{sm\,\mathrm{sgn}(i-j)} & (i\neq j) \end{cases}$$

### Zero-mode

$$P_\lambda, Q_\lambda \quad \left(\lambda \in \bigoplus_{j=1}^{N} \mathbb{Z}\bar{\epsilon}_j \subset \mathbb{C}^{N-1} \subset \mathbb{C}^N\right)$$

$$[iP_\lambda, Q_\mu] = (\lambda|\mu)$$

$\{\epsilon_j\}_{j=1}^N$ orthonormal basis $\quad (\epsilon_i|\epsilon_j)=\delta_{ij}$

$$\bar{\epsilon}_j = \epsilon_j - \frac{1}{N}\sum_{j=1}^{N}\epsilon_j$$

## Bosonization $U_{q,p}(\widehat{sl_N})$

- $$F_j(z) = e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_j}} \left(q^{(\frac{2s}{N}-1)j} z\right)^{\sqrt{\frac{r-1}{r}}P_{\alpha_j} + \frac{r-1}{r}}$$
$$\times \exp\left(-\sum_{m>0} \frac{1}{m} (\beta_{-m}^{j} - \beta_{-m}^{j+1}) (q^{\frac{2s}{N}} z)^{m}\right)$$
$$\times \exp\left(\sum_{m>0} \frac{1}{m} (\beta_{m}^{j} - \beta_{m}^{j+1}) (q^{\frac{2s}{N}} z)^{-m}\right).$$

- $$E_j(z) = e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_j}} \left(q^{(\frac{2s}{N}-1)j} z\right)^{-\sqrt{\frac{r}{r-1}}P_{\alpha_j} + \frac{r}{r-1}}$$
$$\times \exp\left(\sum_{m>0} \frac{1}{m} \frac{[rm]_q}{[(r-1)m]_q} (\beta_{-m}^{j} - \beta_{-m}^{j+1}) (q^{\frac{2s}{N}} z)^{m}\right)$$
$$\times \exp\left(-\sum_{m>0} \frac{1}{m} \frac{[rm]_q}{[(r-1)m]_q} (\beta_{m}^{j} - \beta_{m}^{j+1}) (q^{\frac{2s}{N}} z)^{-m}\right).$$

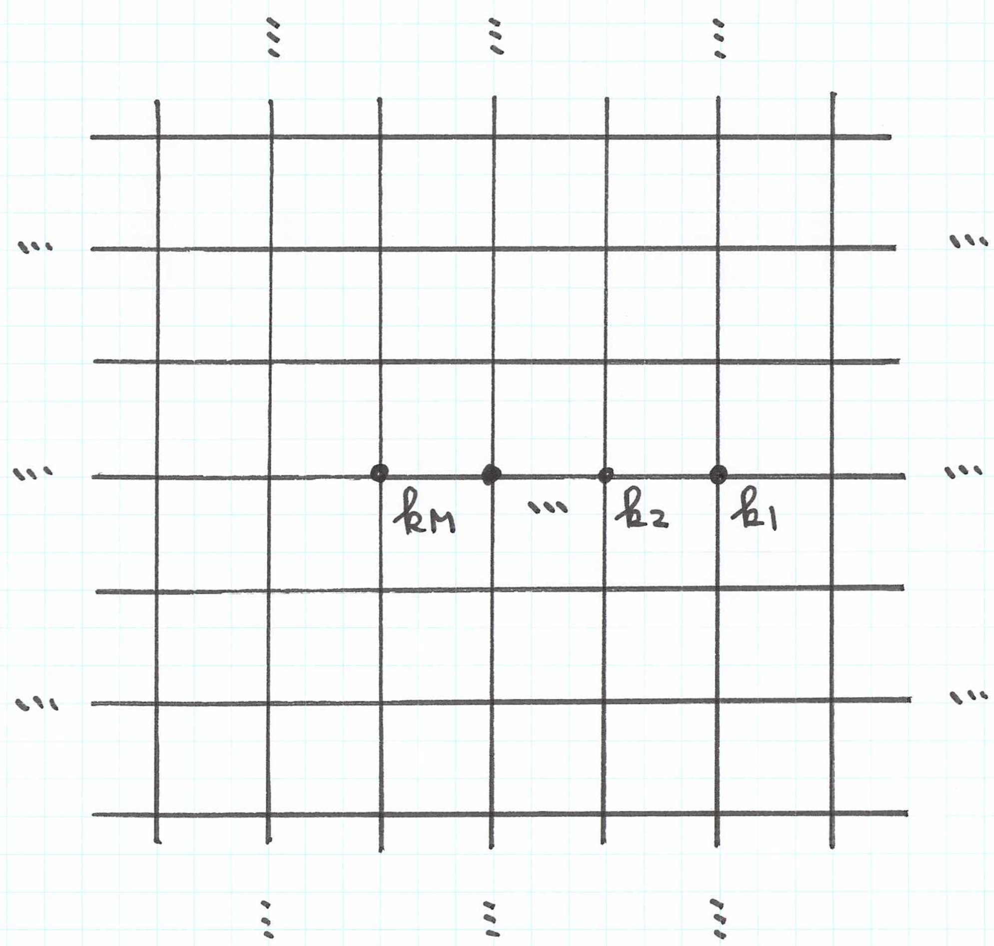
$(j = 1, 2, \cdots, N-1)$

# Boundary $sl_N$ - ABF model

# 2-dim. Lattice model

$m_a$ : Local Variable

$$\text{Boltzmann weight} \quad \begin{array}{ccc} m_a & & m_b \\ & \square & \\ m_c & & m_d \end{array} = W\left(\begin{matrix} m_a & m_b \\ m_c & m_d \end{matrix} \middle| u\right) = \exp\left(-\frac{\mathcal{E}\left(\begin{matrix} m_a & m_b \\ m_c & m_d \end{matrix} \middle| u\right)}{k_B \cdot T}\right)$$




**Probability**

$$P_{k_1 \cdots k_M} = \frac{\sum \prod_j \delta_{k_j, m_j} \prod_{\text{face}} W\left(\begin{matrix} m_a & m_b \\ m_c & m_d \end{matrix} \middle| u\right)}{\sum \prod_{\text{face}} W\left(\begin{matrix} m_a & m_b \\ m_c & m_d \end{matrix} \middle| u\right)}$$

## Yang-Baxter equation

$$\sum_g W\left(\begin{matrix} d & e \\ c & g \end{matrix}\middle| u_1\right) W\left(\begin{matrix} c & g \\ b & a \end{matrix}\middle| u_2\right) W\left(\begin{matrix} e & f \\ g & a \end{matrix}\middle| u_1-u_2\right)$$

$$= \sum_g W\left(\begin{matrix} g & f \\ b & a \end{matrix}\middle| u_1\right) W\left(\begin{matrix} d & e \\ g & f \end{matrix}\middle| u_2\right) W\left(\begin{matrix} d & g \\ c & b \end{matrix}\middle| u_1-u_2\right)$$

## Boltzmann weight

$$W\left(\begin{matrix} a+2\overline{\epsilon_\mu} & a+\overline{\epsilon_\mu} \\ a+\overline{\epsilon_\mu} & a \end{matrix}\middle| u\right) = r(u)$$

$$W\left(\begin{matrix} a+\overline{\epsilon_\mu}+\overline{\epsilon_\nu} & a+\overline{\epsilon_\mu} \\ a+\overline{\epsilon_\nu} & a \end{matrix}\middle| u\right) = r(u)\frac{[u][a_{\mu\nu}-1]}{[u-1][a_{\mu\nu}]}$$

$$W\left(\begin{matrix} a+\overline{\epsilon_\mu}+\overline{\epsilon_\nu} & a+\overline{\epsilon_\nu} \\ a+\overline{\epsilon_\nu} & a \end{matrix}\middle| u\right) = r(u)\frac{[u-a_{\mu\nu}][1]}{[u-1][a_{\mu\nu}]}$$

$a \in \bigoplus_{\mu=1}^{N} \mathbb{Z}\,\overline{\epsilon_\mu} \subset \mathbb{C}^{N-1} \subset \mathbb{C}^N$ , $\overline{\epsilon_\mu} = \epsilon_\mu - \frac{1}{N}\sum_{\nu=1}^{N}\epsilon_\nu$ $\qquad (\epsilon_i|\epsilon_j) = \delta_{ij}$

## $\boxed{\text{Boltzmann weight}}$

(Continuation)

- $r(u) = z^{\frac{r-1}{r}\frac{N-1}{N}} \dfrac{\varphi(1/z)}{\varphi(z)}$

- $\varphi(z) = \dfrac{(q^2 z; q^{2r}, q^{2N})_\infty (q^{2r+2N-2} z; q^{2r}, q^{2N})_\infty}{(q^{2r} z; q^{2r}, q^{2N})_\infty (q^{2N} z; q^{2r}, q^{2N})_\infty}$

- $(z; p_1, p_2)_\infty = \prod_{k_1=0}^{\infty} \prod_{k_2=0}^{\infty} (1 - p_1^{k_1} p_2^{k_2} z)$

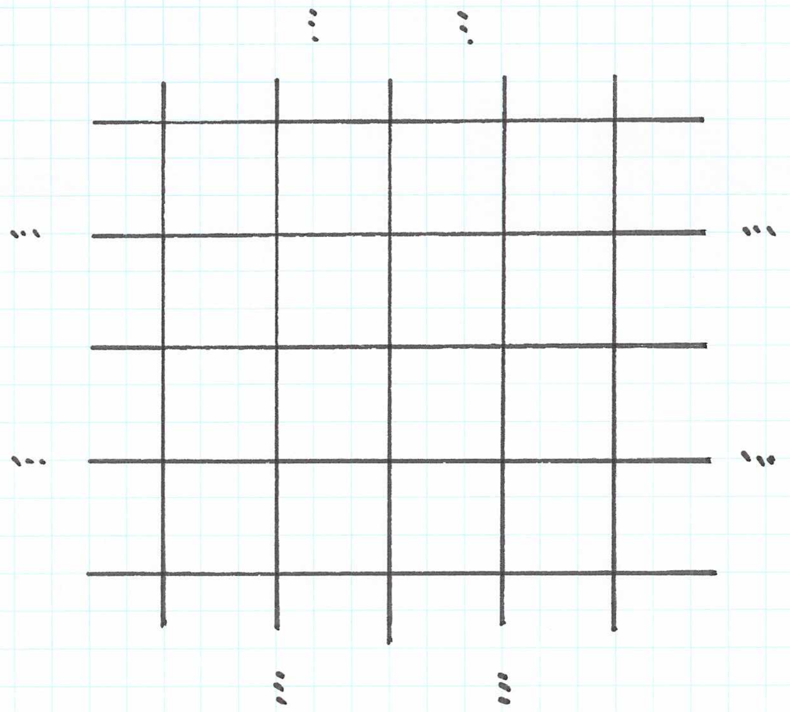
## $\boxed{\text{Restriction}}$

$W\left(\begin{matrix} a & b \\ c & d \end{matrix} \middle| u\right)$

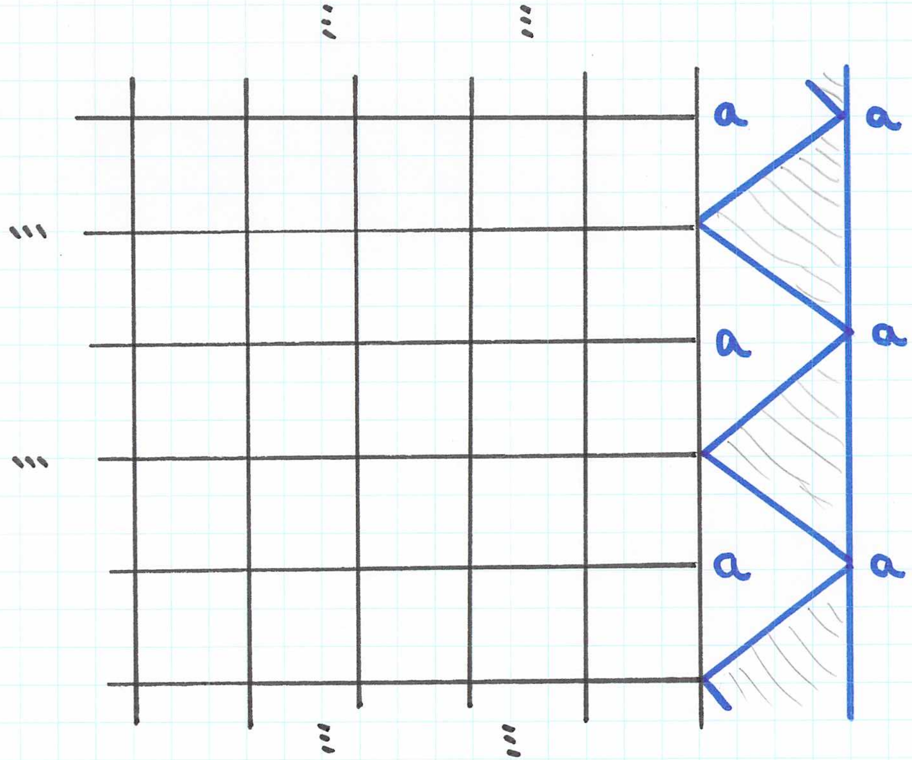
$0 < a_{\mu\nu} < r \Rightarrow [a_{\mu\nu}] \neq 0$

$$a, b, c, d \in \left\{ a \in \bigoplus_{\mu=1}^{N} \mathbb{Z}\bar{\epsilon}_\mu \;\middle|\; \begin{matrix} 0 < a_{\mu\nu} = (a+\rho \mid \bar{\epsilon}_\mu - \bar{\epsilon}_\nu) < r \\ (1 \leq \mu < \nu \leq N) \end{matrix} \right\}$$

# Lattice model

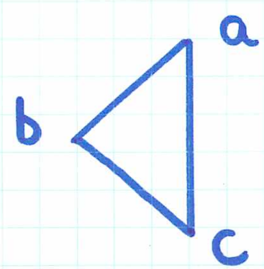


Infinite




Semi-Infinite

# Boundary Boltzmann weight




$$= K\left(b \begin{matrix} a \\ c \end{matrix} \middle| u\right)$$

Boundary
Yang-Baxter equation

## Boundary Yang-Baxter equation

$$\sum_{f,g} W\left(\begin{matrix} c & f \\ b & a \end{matrix}\,\middle|\, u_1-u_2\right) W\left(\begin{matrix} c & d \\ f & g \end{matrix}\,\middle|\, u_1+u_2\right) K\left(f \begin{matrix} g \\ a \end{matrix}\,\middle|\, u_1\right) K\left(d \begin{matrix} e \\ g \end{matrix}\,\middle|\, u_2\right)$$

$$= \sum_{f,g} W\left(\begin{matrix} c & d \\ f & e \end{matrix}\,\middle|\, u_1-u_2\right) W\left(\begin{matrix} c & f \\ b & g \end{matrix}\,\middle|\, u_1+u_2\right) K\left(f \begin{matrix} e \\ g \end{matrix}\,\middle|\, u_1\right) K\left(b \begin{matrix} g \\ a \end{matrix}\,\middle|\, u_2\right)$$

## Boundary Boltzmann weight

$$K\left(a+\bar{\epsilon}_\mu \begin{matrix} a \\ b \end{matrix}\,\middle|\, u\right) = z^{\frac{r-1}{r}\frac{N-1}{N} - \frac{2}{r}a_1} \frac{h(z)}{h(1/z)} \frac{[c-u][a_{1,\mu}+c+u]}{[c+u][a_{1,\mu}+c-u]} \delta_{ab}$$

Boudary Boltzmann weight

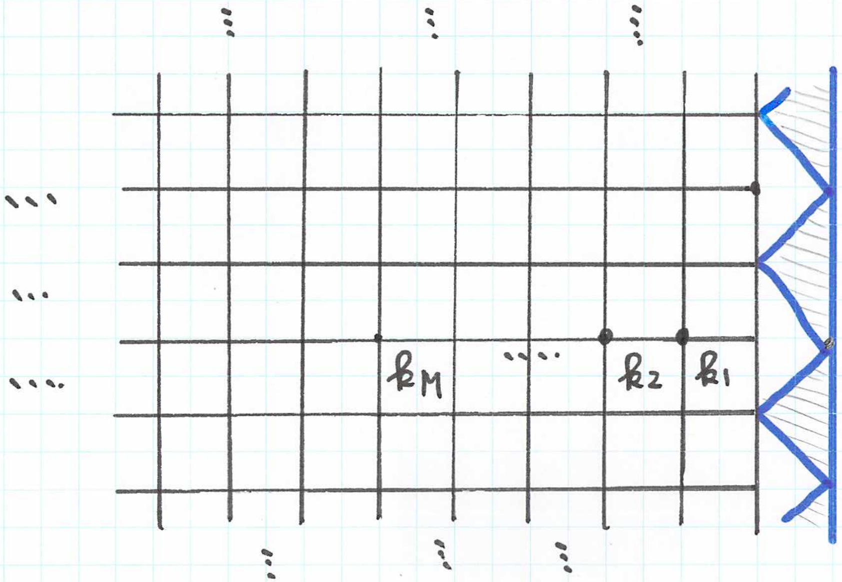
$$h(z) = \frac{(q^{2r+2N-2}/z^2 ; q^{2r}, q^{4N})_\infty (q^{2N+2}/z^2 ; q^{2r}, q^{4N})_\infty}{(q^{2r}/z^2 ; q^{2r}, q^{4N})_\infty (q^{4N}/z^2 ; q^{2r}, q^{4N})_\infty}$$

$$\times \frac{(q^{2N+2c}/z ; q^{2r}, q^{2N})_\infty (q^{2r-2c}/z ; q^{2r}, q^{2N})_\infty}{(q^{2N+2r-2c-2}/z ; q^{2r}, q^{2N})_\infty (q^{2c+2}/z ; q^{2r}, q^{2N})_\infty}$$

$$\times \prod_{j=2}^{N} \frac{(q^{2r+2N-2c-2a_{1j}}/z ; q^{2r}, q^{2N})_\infty (q^{2c+2a_{1j}}/z ; q^{2r}, q^{2N})_\infty}{(q^{2r+2N-2c-2a_{1j}-2}/z ; q^{2r}, q^{2N})_\infty (q^{2c+2+2a_{1j}}/z ; q^{2r}, q^{2N})_\infty}$$

## $sl_N$ - ABF model

$U_{q,p}(\widehat{sl_N})$ $\qquad q, \ p = q^{2r}$




$\boxed{N = 2, 3, 4, \cdots \text{ and } r \geq N+2 \ (r \in \mathbb{Z})}$

Ex. $N = 2, \ r = 4$
2 dimensional Ising

Ex. $N = 2, \ r \to \infty$
XXZ chain

## Correlation function (Probability)

$$P_{k_1 k_2 \cdots k_M} = \frac{\sum \prod_j \delta_{k_j m_j} \prod_{\text{face}} W\left(\begin{matrix} m_a & m_b \\ m_c & m_d \end{matrix} \middle| u\right) \prod_{\text{boundary}} K\left(m_e \begin{matrix} m_f \\ m_g \end{matrix} \middle| u\right)}{\sum \prod_{\text{face}} W\left(\begin{matrix} m_a & m_b \\ m_c & m_d \end{matrix} \middle| u\right) \prod_{\text{boundary}} K\left(m_e \begin{matrix} m_f \\ m_g \end{matrix} \middle| u\right)}$$

$W, K$ : elliptic

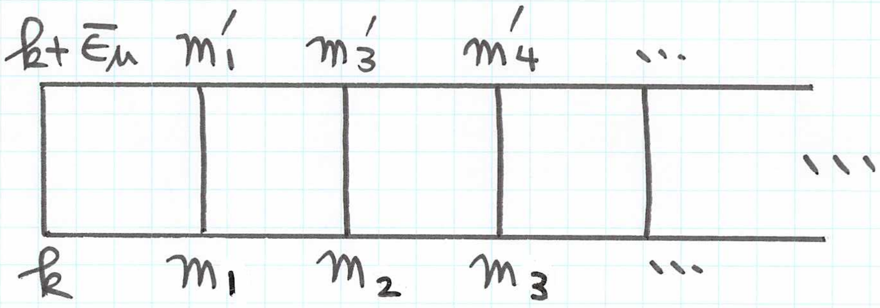
# Vertex Operator Approach

- Correlation Function

Physics

## Vertex operator

$$\left[\hat{\Phi}_\mu(u)\right]_m^{m'} =$$




$$= \prod_{j=1}^{\infty} W\left(\begin{matrix} m'_j & m'_{j+1} \\ m_j & m_{j+1} \end{matrix} \middle| u\right) \qquad \left(\begin{matrix} m'_1 = k+\bar{\epsilon}_\mu \\ m_1 = k \end{matrix}\right)$$

## Commutation relation

$$\hat{\Phi}_{\mu_1}(u_1)\hat{\Phi}_{\mu_2}(u_2) = \sum_{\varepsilon_{\mu_1}+\varepsilon_{\mu_2} = \varepsilon_{\mu'_1}+\varepsilon_{\mu'_2}} W\left(\begin{matrix} k+\bar{\varepsilon}_{\mu_1}+\bar{\varepsilon}_{\mu_2} & k+\bar{\varepsilon}_{\mu'_1} \\ k+\bar{\varepsilon}_{\mu_2} & k \end{matrix} \middle| u_1-u_2\right)$$

$$\times \hat{\Phi}_{\mu'_2}(u_2)\hat{\Phi}_{\mu'_1}(u_1)$$

## Vertex operator

**Mathematics**

$$\Phi(u) : V(\lambda) \longrightarrow V_z \otimes V(\mu)$$

$$\Delta(x) \cdot \Phi(u) = \Phi(u) \cdot x \,, \qquad x \in U_{q,p}(\widehat{sl_N})$$

Hopf algebroid

**Intertwing Property**

## Commutation relation

$$\Phi(u) = \sum_{\mu=1}^{N} v_\mu \Phi_\mu(u)$$

$$\Phi_{\mu_1}(u_1)\Phi_{\mu_2}(u_2) = \sum_{\varepsilon_{\mu_1}+\varepsilon_{\mu_2}=\varepsilon_{\mu_1'}+\varepsilon_{\mu_2'}} W\left(\begin{matrix} k+\bar{\varepsilon}_{\mu_1}+\bar{\varepsilon}_{\mu_2} & k+\bar{\varepsilon}_{\mu_1'} \\ k+\bar{\varepsilon}_{\mu_2} & k \end{matrix} \middle| u_1-u_2\right)$$

$$\times \Phi_{\mu_2'}(u_2)\Phi_{\mu_1'}(u_1).$$

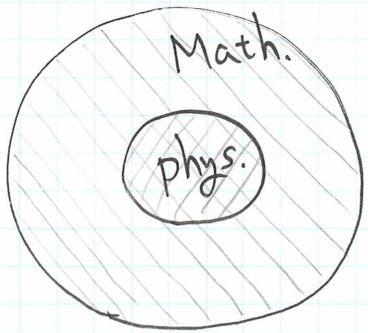
Ansatz

Vertex Operator Approach

$$\hat{\Phi}_\mu(u) = \Phi_\mu(u)$$

Physics Mathematics

Basis




- the same commutation relations
- the same character of the space

etc.

## Bosonization of Vertex operator

$U_{q,p}(\widehat{sl_N})$

$$\Phi_\mu(u) = \int \cdots \int_C \prod_{j=1}^{\mu-1} \frac{dz_j}{z_j} \Phi_0(u) F_1(u_1) F_2(u_2) \cdots F_\mu(u_\mu)$$

$$\times \prod_{j=1}^{\mu-1} \frac{\left[u_j - u_{j-1} - \sqrt{r(r-1)}\, P_{\bar{\epsilon}_\mu}\right]}{\left[u_j - u_{j-1} - \frac{1}{2}\right]}$$

$$\left( \begin{array}{l} z_j = q^{2u_j} \\ u_0 = u \end{array} \right)$$

$$\Phi_0(u) = z^{\frac{r-1}{2r}\frac{N-1}{N}} e^{-i\sqrt{\frac{r-1}{r}} Q_{\bar{\epsilon}_1}} z^{-\sqrt{\frac{r-1}{r}} P_{\bar{\epsilon}_1}}$$

$$\times \exp\left( -\sum_{m>0} \frac{1}{m} \left(B_{-m}^{j} - B_{-m}^{j+1}\right) q^{jm} z^{m} \right)$$

$$\times \exp\left( \sum_{m>0} \frac{1}{m} \left(B_{m}^{j} - B_{m}^{j+1}\right) q^{-jm} z^{-m} \right)$$

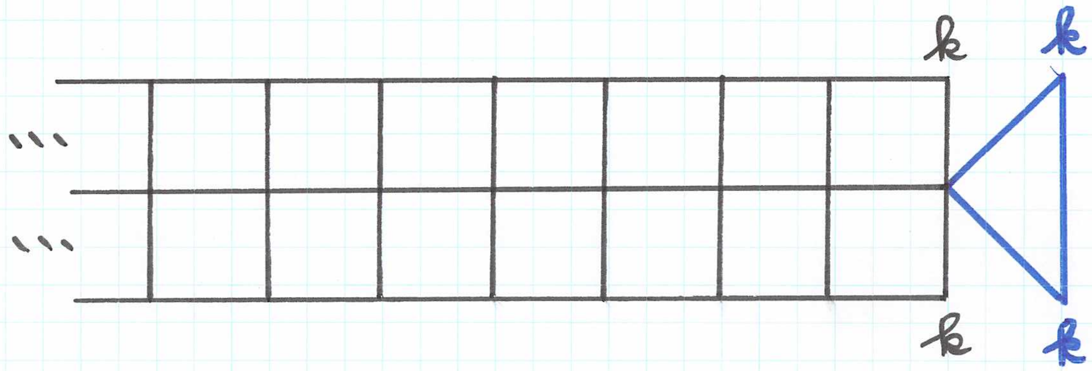
[ Asai, Jimbo, Miwa, Pugai ] (1996)

# Transfer matrix

2 dim Classical Mechanics
= 1 dim Quantum Mechanics

$$T_B(u) = \sum_{\mu=1}^{N} \Phi_\mu^*(u)\, K\left(k+\bar{\epsilon}_\mu \,{}^{k}_{k} \middle| u\right) \Phi_\mu(-u)$$

=



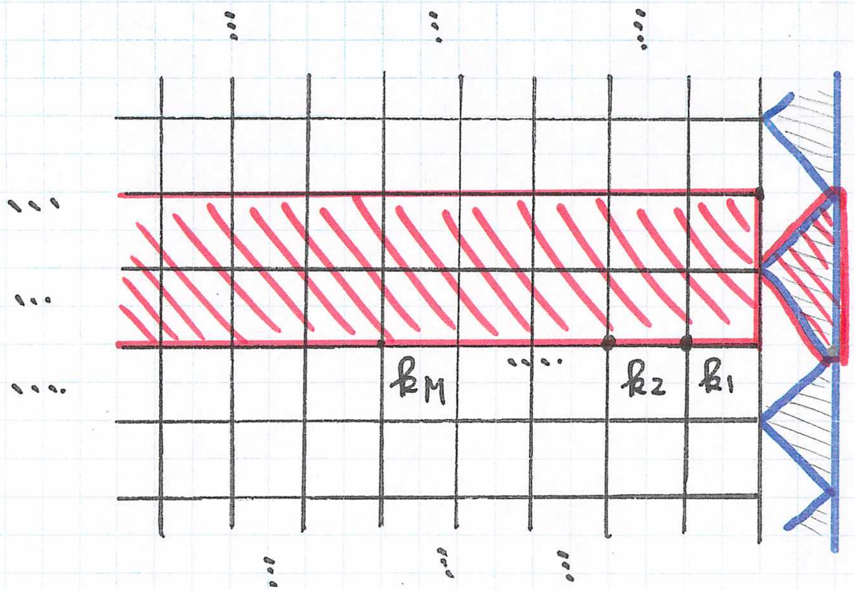

**Diagonalize $T_B(u)$ !**

$$[T_B(u_1), T_B(u_2)] = 0 \quad (\forall u_1, u_2)$$

## $sl_N$ - ABF model

$U_{q,p}(\widehat{sl_N})$ $\quad q,\ p = q^{2r}$

$N = 2, 3, 4, \cdots$ and $r \geq N+2$ $(r \in \mathbb{Z})$




Ex. $N=2,\ r=4$
2 dimensional Ising

Ex. $N=2,\ r \to \infty$
XXZ chain

## Correlation function (Probability)

$$P_{k_1 k_2 \cdots k_M} = \frac{\sum \prod_j \delta_{k_j m_j} \prod_{\text{face}} W\left(\begin{matrix} m_a & m_b \\ m_c & m_d \end{matrix} \middle| u\right) \prod_{\text{boundary}} K\left(m_e \begin{matrix} m_f \\ m_g \end{matrix} \middle| u\right)}{\sum \prod_{\text{face}} W\left(\begin{matrix} m_a & m_b \\ m_c & m_d \end{matrix} \middle| u\right) \prod_{\text{boundary}} K\left(m_e \begin{matrix} m_f \\ m_g \end{matrix} \middle| u\right)}$$

**$W, K$ : elliptic**

## Linear algebra for undergraduate

- $A, B \in M(n)$ : Diagonalizable
- $[A, B] = 0$

$\Longrightarrow$ $A, B$ : **Simultaneously** Diagonalizable

$$P^{-1} A P = \begin{pmatrix} \alpha_1 & & O \\ & \ddots & \\ O & & \alpha_n \end{pmatrix}, \quad P^{-1} B P = \begin{pmatrix} \beta_1 & & O \\ & \ddots & \\ O & & \beta_n \end{pmatrix}$$

The eigen-vectors don't depend on spectral parameter $u$

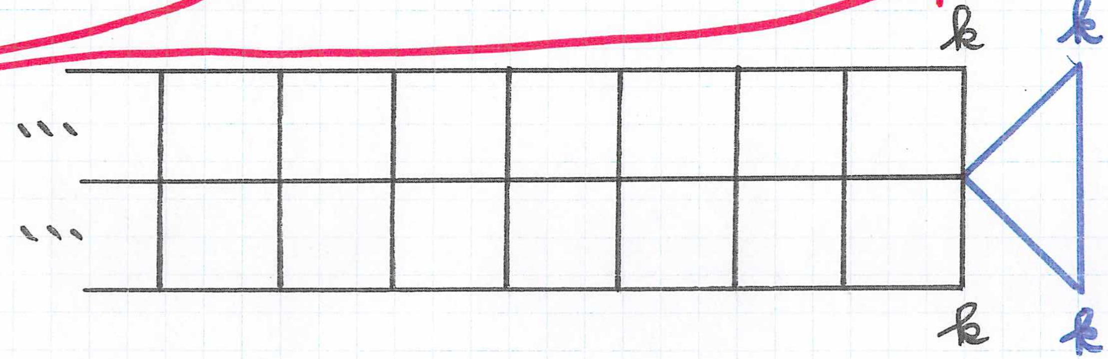
$$[T_B(u_1), T_B(u_2)] = 0$$

## Transfer matrix

2 dim Classical Mechanics
= 1 dim Quantum Mechanics

$$T_B(u) = \sum_{\mu=1}^{N} \Phi_\mu^*(u)\, K\left(k+\bar{\epsilon}_\mu \,{}^{k}_{k} \middle| u\right) \Phi_\mu(-u)$$

boson =



Find Bosonization of eigenvectors.

$$[T_B(u_1), T_B(u_2)] = 0 \quad (\forall u_1, u_2)$$

# Boundary Vacuum state

$U_{q,p}(\widehat{sl_N})$

[Kojima] (2011)

- $T_B(u)|k\rangle_B = |k\rangle_B$

- $|k\rangle_B = \exp(B)\exp\left(\frac{\sqrt{-1}}{\sqrt{r(r-1)}}Q_k\right)|0\rangle$

- $$B = -\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j,k=1}^{N-1}\frac{1}{m}\frac{[rm]_q}{[(r-1)m]_q} I_{j,k}(m)\alpha_{-m}^{j}\alpha_{-m}^{k} + \sum_{m=1}^{\infty}\sum_{j=1}^{N-1}\frac{1}{m}D_j(m)\beta_m^{j}$$

- $$I_{jk}(m) = I_{kj}(m) = \frac{[jm]_q[(N-k)m]_q}{[m]_q[Nm]_q} \quad (j<k)$$

- $\alpha_m^{j} = q^{-jm}(\beta_m^{j} - \beta_m^{j+1})$

## $\boxed{\text{Boundary Vacuum State}}$ (continuation)

$$\boxed{\begin{aligned} D_{\bar{j}}(m) = &- \Theta_m\left( \frac{[(N-\bar{j})m/2]_q \{hm/2\}_q}{[(h-1)m/2]_q} q^{\frac{(3\bar{j}-N-1)m}{2}} \right) \\ &+ q^{(\bar{j}-1)m} \frac{[(-h+2\pi_{1\bar{j}}+2c-\bar{j}+2)m]_q}{[(h-1)m]_q} \\ &+ q^{(h-2c+2\bar{j}-2)m} \frac{[m]_q}{[(h-1)m]_q} \left( \sum_{k=\bar{j}+1}^{N-1} q^{-2m\pi_{1k}} \right) \\ &+ q^{(2\bar{j}-N)m} \frac{[(h-2\pi_{1N}-2c+N-1)m]_q}{[(h-1)m]_q} \end{aligned}}$$

$$\{a\}_q = q^a + q^{-a}, \quad \Theta_m(x) = \begin{cases} x & m: \text{even} \\ 0 & m: \text{odd} \end{cases}$$

## Diagonalization of $T_B(u)$

Excited States

$$\Psi^*_{\mu_1}(v_1)\Psi^*_{\mu_2}(v_2)\cdots\Psi^*_{\mu_s}(v_N)|k\rangle_B$$

Eigen-vectors $(\mu_j=1,2,\cdots,N;\ s=0,1,2,\cdots)$

## Type-II Vertex Operators

$$\Phi_\varepsilon(u)\Psi^*_\mu(v)=\chi(u-v)\Psi^*_\mu(v)\Phi_\varepsilon(u)$$

$$\chi(u)=z^{\frac{1-N}{N}}\frac{(q^{2N-1}/z;q^{2N})_\infty(qz;q^{2N})_\infty}{(q/z;q^{2N})_\infty(q^{2N-1}z;q^{2N})_\infty}\qquad(z=q^{2u})$$

## Type-II Vertex Operators

$U_{q,p}(\widehat{sl_N})$

$$\Psi_\mu^*(u) = \int \cdots \int_C \prod_{j=1}^{\mu-1} \frac{dz_j}{z_j} \Psi_0^*(u) E_1(u_1) E_2(u_2) \cdots E_\mu(u_\mu)$$

$$\times \prod_{j=1}^{\mu-1} \frac{[u_j - u_{j-1} - \frac{1}{2} + \sqrt{r(r-1)} P_{\epsilon_\mu}]^*}{[u_j - u_{j-1} + \frac{1}{2}]^*} \qquad \left( \begin{array}{c} z_j = q^{2u_j} \\ u_0 = u \end{array} \right)$$

$$\Psi_0^*(u) = z^{\frac{r}{2(r-1)}\frac{N-1}{N}} e^{i\sqrt{\frac{r}{r-1}} Q_{\bar{\epsilon}_1}} z^{\sqrt{\frac{r}{r-1}} P_{\bar{\epsilon}_1}}$$

$$\times \exp\left( \sum_{m>0} \frac{1}{m} \frac{[rm]_q}{[(r-1)m]_q} (B_{-m}^j - B_{-m}^{j+1}) q^{jm} z^m \right)$$

$$\times \exp\left( - \sum_{m>0} \frac{1}{m} \frac{[rm]_q}{[(r-1)m]_q} (B_m^j - B_m^{j+1}) q^{-jm} z^{-m} \right).$$

[ Furutsu, Kojima, Quano ] (2000)

Elliptic algebra $U_{q,p}(\widehat{sl_N})$

vertex operator

- $[u_1-u_2+1]\, F_{\bar{j}}(z_1) F_{\bar{j}}(z_2) = [u_1-u_2-1]\, F_{\bar{j}}(z_2) F_{\bar{j}}(z_1)$
- $[u_1-u_2-1]^*\, E_{\bar{j}}(z_1) E_{\bar{j}}(z_2) = [u_1-u_2+1]^*\, E_{\bar{j}}(z_2) E_{\bar{j}}(z_1)$

type II vertex operator

- $[u] = q^{\frac{u^2}{r}-u}\, \Theta_{q^{2r}}(q^{2u})$ $\qquad r>0$
- $[u]^* = q^{\frac{u^2}{r^*}-u}\, \Theta_{q^{2r^*}}(q^{2u})$ $\qquad r^* = r-k>0$

2-periods

## Type-II Vertex operator

$$\Psi^*(u) : V(\lambda) \longrightarrow V(\mu) \otimes V_z$$

$$\Delta(x) \cdot \Psi^*(u) = \Psi^*(u) \cdot x \quad , \quad x \in U_{qp}(\widehat{sl_2})$$

Intertwing Property

| Correlation function | Probability |
|---|---|

- $P_{k_1 k_2 \cdots k_M} = P_{k_1 k_2 \cdots k_M}(z_1, z_2, \cdots, z_M)\big|_{z_1 = \cdots = z_M = z}$ .

- $P_{k_1 k_2 \cdots k_M}(z_1, z_2, \cdots, z_M)$

$$= \frac{{}_B\langle k | \Phi^*_{\varepsilon_{\mu_1}}(z_1) \cdots \Phi^*_{\varepsilon_{\mu_M}}(z_M) \Phi_{\varepsilon_{\mu_M}}(z_M) \cdots \Phi_{\varepsilon_{\mu_1}}(z_1) | k \rangle_B}{{}_B\langle k | k \rangle_B} .$$

Here $k_{j+1} = k_j + \bar{\varepsilon}_{\mu_j}$ , $k_1 = k$ .

Everything has a bosonization.

## Correlation function

$N=2 \quad \{z\}_\infty = (z;q^4,q^4,q^{2r})_\infty , \quad A=\{j \mid \varepsilon_j = -\}$

$${}_B\langle k | \Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2)\cdots\Phi_{\varepsilon_M}(z_M)|k\rangle_B \Big/ {}_B\langle k|k\rangle_B$$

$$= q^{\frac{r-1}{r}|A|}\left(\frac{(q^2;q^2)_\infty}{(q^{2r-2};q^{2r})_\infty}\right)^{|A|}\left(\frac{(q^6;q^4,q^{2r})_\infty}{(q^{4+2r};q^4,q^{2r})_\infty}\right)^{2|A|}\left(\frac{\{q^6\}_\infty\{q^{2r+6}\}_\infty}{\{q^8\}_\infty\{q^{2r+4}\}_\infty}\right)^M$$

$$\times\prod_{j=1}^M z_j^{-\frac{k}{2r}}\prod_{1\le j<k\le M}\frac{\{q^6 z_j z_k\}_\infty\{q^{2r+6}z_j z_k\}_\infty\{q^2/z_j z_k\}_\infty\{q^{2r+2}/z_j z_k\}_\infty}{\{q^8 z_j z_k\}_\infty\{q^{2r+4}z_j z_k\}_\infty\{q^4/z_j z_k\}_\infty\{q^{2r}/z_j z_k\}_\infty}$$

$$\times\prod_{1\le j<k\le M} z_j^{\frac{r-1}{2r}}\frac{\{q^6 z_j/z_k\}_\infty\{q^{2r+6}z_j/z_k\}_\infty\{q^2 z_k/z_j\}_\infty\{q^{2r+2}z_k/z_j\}_\infty}{\{q^8 z_j/z_k\}_\infty\{q^{2r+4}z_j/z_k\}_\infty\{q^4 z_k/z_j\}_\infty\{q^{2r}z_k/z_j\}_\infty}$$

$$\times\prod_{j=1}^M\frac{(q^{10}z_j^2;q^4,q^8,q^{2r})_\infty(q^{2r+6}z_j^2;q^4,q^8,q^{2r})_\infty(q^6/z_j^2;q^4,q^8,q^{2r})_\infty(q^{2r+2}/z_j^2;q^4,q^4,q^{2r})_\infty}{(q^8 z_j^2;q^4,q^8,q^{2r})_\infty(q^{2r+4}z_j^2;q^4,q^8,q^{2r})_\infty(q^8/z_j^2;q^4,q^8,q^{2r})_\infty(q^{2r}/z_j^2;q^4,q^4,q^{2r})_\infty}$$

$$\times\prod_{j=1}^M\frac{(q^{2r-2c-2k+2}z_j;q^4,q^{2r})_\infty(q^{2c+2}z_j;q^4,q^{2r})_\infty(q^{2r-2c}/z_j;q^4,q^{2r})_\infty(q^{2c+2k}/z_j;q^4,q^{2r})_\infty}{(q^{2r-2c-2k+4}z_j;q^4,q^{2r})_\infty(q^{2c+4}z_j;q^4,q^{2r})_\infty(q^{2r-2c+2}/z_j;q^4,q^{2r})_\infty(q^{2c+2k+2}/z_j;q^4,q^{2r})_\infty}$$

......

$$\times \oint \cdots \oint_C \prod_{a\in A} \frac{dw_a}{2\pi i w_a} \prod_{a\in A} \frac{\left[u_a - v_a + \frac{1}{2} - k - \sum_{j=a+1}^{N} \varepsilon_j\right]}{\left[u_a - v_a + \frac{1}{2}\right]} w_a^{\frac{k}{h}} \frac{(q^6 w_a^2; q^4, q^{2h})_\infty (q^2/w_a^2; q^4, q^{2h})_\infty}{(q^{2h+2} w_a^2; q^4, q^{2h})_\infty (q^{2h}/w_a^2; q^4, q^{2h})_\infty}$$

$$\times \prod_{j=1}^{M} \prod_{a\in A} \frac{(q^{2h+3} z_j w_a; q^4, q^{2h})_\infty (q^{2h-1}/z_j w_a; q^4, q^{2h})_\infty}{(q^5 z_j w_a; q^4, q^{2h})_\infty (q/z_j w_a; q^4, q^{2h})_\infty}$$

$$\times \prod_{j=1}^{M} \prod_{a\in A} w_a^{-\frac{h-1}{h}} \frac{(q^{2h-1} z_j/w_a; q^4, q^{2h})_\infty (q^{2h+3} w_a/z_j; q^4, q^{2h})_\infty}{(q z_j/w_a; q^4, q^{2h})_\infty (q^5 w_a/z_j; q^4, q^{2h})_\infty}$$

$$(w_a = q^{2v_a})$$

[Kojima]

$$\times \prod_{j=1}^{M} \prod_{a\in A} z_j^{-\frac{h-1}{h}} \frac{(q^{2h+3} z_j/w_a; q^4, q^{2h})_\infty (q^{2h-1} w_a/z_j; q^4, q^{2h})_\infty}{(q^5 z_j/w_a; q^4, q^{2h})_\infty (q w_a/z_j; q^4, q^{2h})_\infty}$$

$$\times \prod_{a\in A} \frac{1}{(q^{2h-2c-2k+1} w_a; q^{2h})_\infty (q^{2c+1} w_a; q^{2h})_\infty (q^{2h-2c-1}/w_a; q^{2h})_\infty (q^{2c+2k-1}/w_a; q^{2h})_\infty}$$

$$\times \prod_{\substack{a,b\in A \\ a<b}} \frac{(q^4 w_a w_b; q^2, q^{2h})_\infty (1/w_a w_b; q^2, q^{2h})_\infty (q^4 w_a/w_b; q^2, q^{2h})_\infty (q^4 w_b/w_a; q^2, q^{2h})_\infty}{(q^{2h+2} w_a w_b; q^2, q^{2h})_\infty (q^{2h-2}/w_a w_b; q^2, q^{2h})_\infty (q^{2h+2} w_a/w_b; q^2, q^{2h})_\infty (q^{2h+2} w_b/w_a; q^2, q^{2h})_\infty} .$$

## Specialization

$U_{qp}(\widehat{sl_2}) \longrightarrow U_q(\widehat{sl_2}) \qquad (-1<q<0)$

$$q^{2r} = p \longmapsto 0$$
$$r \longmapsto \infty$$

## XXZ chain

$$H_B = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma_{k+1}^x\sigma_k^x + \sigma_{k+1}^y\sigma_k^y + \Delta\,\sigma_{k+1}^z\sigma_k^z\right) + h\sigma_1^z.$$

$$\left(\Delta = \frac{q+q^{-1}}{2}, \quad h = \frac{1-q^2}{-4q}\,\frac{1+s}{1-s}\right)$$

## Correlation function

- $$\frac{{}_B\langle 1|\sigma_1^z|1\rangle_B}{{}_B\langle 1|1\rangle_B} = 1 + 2\sum_{l=1}^{\infty}\frac{(-q^2)^l(1-1/s)^2}{(1-q^{2l}/s)^2}$$

$$\left.\frac{{}_B\langle 1|\sigma_1^z|1\rangle_B}{{}_B\langle 1|1\rangle_B}\right|_{s=-1} = \frac{(q^2;q^2)_\infty^4}{(-q^2;q^2)_\infty^4}$$

Dirichlet boundary

$h=0$

[Jimbo, Kedem, Kojima, Konno, Miwa] (1995)

- We have studied elliptic deformation of XXZ chain.
- Next we study another generalization of XXZ chain.

Boundary condition

(2) Triangular Boundary XXZ chain

- Correlation function
- Spin-Reversal Property

$\boxed{\text{Triangular XXZ chain}}$

# Triangular Boundary XXZ chain

$$H_B^{(-)} = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma_{k+1}^x\sigma_k^x + \sigma_{k+1}^y\sigma_k^y + \Delta\sigma_{k+1}^z\sigma_k^z\right) + h\sigma_1^z + k\sigma^-.$$

$$\sigma^z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma^- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \qquad h, k \in \mathbb{R}$$

$$H_B \curvearrowright \mathcal{H}^{(i)} \subset \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

Semi-Infinite Chain

$$|p\rangle = \cdots \otimes v_{p(s)} \otimes \cdots \otimes v_{p(2)} \otimes v_{p(1)}$$

$\mathcal{H}^{(i)}$ is spanned by $|p\rangle$ s.t. $p(s) = (-1)^{s+i}$ $\quad s \gg 1$

$(i = 0, 1)$

## R-matrix

$U_q(\widehat{sl_2})$

$$R(\xi) = \frac{1}{K(\xi)} \begin{array}{c|cccc|} & ++ & +- & -+ & -- \\ \hline ++ & 1 & & & \\ +- & & \frac{(1-\xi^2)q}{1-q^2\xi^2} & \frac{(1-q^2)\xi}{1-q^2\xi^2} & \\ -+ & & \frac{(1-q^2)\xi}{1-q^2\xi^2} & \frac{(1-\xi^2)q}{1-q^2\xi^2} & \\ -- & & & & 1 \end{array} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

$$K(\xi) = \xi \frac{(q^4\xi^2; q^4)_\infty (q^2/\xi^2; q^4)_\infty}{(q^4/\xi^2; q^4)_\infty (q^2\xi^2; q^4)_\infty}$$

## Yang-Baxter equation (YBE)

$$R_{12}(\xi_1/\xi_2) R_{13}(\xi_1/\xi_3) R_{23}(\xi_2/\xi_3)$$
$$= R_{23}(\xi_2/\xi_3) R_{13}(\xi_1/\xi_3) R_{12}(\xi_1/\xi_2)$$

## K-matrix

$$K^{(-)}(\xi) = \frac{\varphi(\xi^2)}{\varphi(\xi^{-2})} \left[\begin{array}{c|c} \dfrac{1-r\xi^2}{\xi^2-r} & 0 \\ \hline \dfrac{s\xi(\xi^2-\xi^{-2})}{\xi^2-r} & 1 \end{array}\right] \quad \left( \begin{array}{l} h = \dfrac{1-q^2}{-4q}\dfrac{1+r}{1-r} \\ k = -\dfrac{s}{1-r} \end{array} \right)$$

$$\varphi(z) = \frac{(q^4 r z ; q^4)_\infty (q^6 z^2 ; q^8)_\infty}{(q^2 r z ; q^4)_\infty (q^8 z^2 ; q^8)_\infty}$$

## Boundary Yang-Baxter equation

$$K_2^{(-)}(\xi_2) R_{21}(\xi_1\xi_2) K_1^{(-)}(\xi_1) R_{12}(\xi_1/\xi_2)$$

$$= R_{21}(\xi_1/\xi_2) K_1^{(-)}(\xi_1) R_{12}(\xi_1\xi_2) K_2^{(-)}(\xi_2)$$

## Transfer mathix

$$T_B^{(i,-)}(\zeta) = \sum_{\bar{j}=\pm} \hat{\Phi}^{*(i,1-i)}_{\bar{j}}(\zeta) K^{(-)}(\zeta)^{\bar{j}}_{\bar{j}} \hat{\Phi}^{(1-i,i)}_{\bar{j}}(\zeta^{-1})$$

$$\left(\hat{\Phi}^{(1-i,i)}_{\bar{j}_0}(\zeta)\right)^{\cdots k_2' k_1'}_{\cdots k_2 k_1} = \sum_{\bar{j}_1,\bar{j}_2\cdots=\pm} \prod_{s=1}^{\infty} R(\zeta)^{\bar{j}_s\, k_s'}_{\bar{j}_{s-1}\, k_s}$$

$$\left(\hat{\Phi}^{*(1-i,i)}_{\bar{j}}(\zeta)\right)^{\cdots k_2' k_1'}_{\cdots k_2 k_1} = \sum_{\bar{j}_1,\bar{j}_2\cdots=\pm} \prod_{s=1}^{\infty} R(\zeta)^{k_s'\, \bar{j}_{s-1}}_{k_s\, \bar{j}_s}$$

$$\left(\begin{array}{c} k_N = (-1)^{N+i} \\ N \gg 1 \end{array}\right)$$

$$\frac{d}{d\zeta} T_B^{(i,-)}(\zeta)\Big|_{\zeta=1} = \frac{4q}{1-q^2} H_B^{(i,-)} + \text{const.}$$

$$H_B^{(i,-)} = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma^x_{k+1}\sigma^x_k + \sigma^y_{k+1}\sigma^y_k + \Delta\sigma^z_{k+1}\sigma^z_k\right) + h\sigma_1^z + k\sigma_1^- \,)$$

$$[T_B^{(i,-)}(\zeta_1), T_B^{(i,-)}(\zeta_2)] = 0 \qquad (\forall \zeta_1, \zeta_2)$$

$\boxed{\text{Vertex Operator Approach}}$

Correlation Function

# Vertex Operator Approach

| Physics | | Mathematics |
|---|---|---|
| $\hat{\Phi}_{\bar{j}}^{(1-\bar{i},\bar{i})}(\xi)$ | $=$ | $\Phi_{\bar{j}}^{(1-\bar{i},\bar{i})}(\xi)$ |
| $\mathcal{H}^{(\bar{i})}$ | $=$ | $V(\Lambda_{\bar{i}})$ |

$\boxed{\text{Ansatz}}$

$$\begin{cases} \left(\hat{\Phi}_{\bar{j}}^{(1-\bar{i},\bar{i})}(\xi)\right)^{\cdots k_N' \cdots k_1'}_{\cdots k_N \cdots k_1} = \lim_{N\to\infty} \sum_{j_1 \cdots j_N = \pm} \prod_{s=1}^{N} R(\xi)^{j_s \; k_s'}_{j_{s-1} \; k_s} \\ \mathcal{H}^{(\bar{i})} \subset \cdots V^{\otimes\infty} \end{cases}$$

$$\begin{cases} \Phi_{\bar{j}}^{(1-\bar{i},\bar{i})}(\xi) : V(\Lambda_{\bar{i}}) \longrightarrow V(\Lambda_{1-\bar{i}}) \otimes V_\xi \\ V(\Lambda_{\bar{i}}) : \text{Irreducible Level } k=1 \text{ highest weight representation} \end{cases}$$

# Bosonization

$U_q(\widehat{sl_2})$, Vertex Operator

## Boson

$$[a_m, a_n] = \delta_{m+n,0}\frac{[m]_q[2m]_q}{m} \qquad (m,n\in\mathbb{Z}_{\neq 0})$$

$$[\partial,\alpha] = 2$$

## Vertex Operator

$$\Phi_-^{(1-i,i)}(\xi) = e^{P(\xi^2)} e^{Q(\xi^2)} e^{\frac{\alpha}{2}} (-q^3\xi^2)^{\frac{\partial+i}{2}} \xi^{-i}$$

$$\Phi_+^{(1-i,i)}(\xi) = \oint_C \frac{dw}{2\pi\sqrt{-1}} \frac{(1-q^2)w\xi}{q(w^2-q^2\xi^2)(w-q^4\xi^2)} :\Phi_-^{(1-i,i)}(\xi)X^-(w):$$

## Basic Operator

$$P(z) = \sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]_q} q^{\frac{7}{2}n} z^n \quad , \quad R^{\pm}(w) = \pm\sum_{n=1}^{\infty}\frac{a_{-n}}{[n]_q} q^{\mp\frac{n}{2}} w^n$$

$$Q(z) = \sum_{n=1}^{\infty}\frac{a_n}{[2n]_q} q^{-\frac{5}{2}n} z^{-n} \quad , \quad S^{\pm}(w) = \mp\sum_{n=1}^{\infty}\frac{a_n}{[n]_q} q^{\mp\frac{n}{2}} w^{-n}$$

$$X^{\pm}(w) = e^{R^{\pm}(w)} e^{S^{\pm}(w)} e^{\pm\alpha} w^{\pm\partial}$$

## Transfer matrix

Find eigenvector of transfer matrix

$$T_B^{(\bar{i},-)}(\zeta) = \sum_{\bar{j}=\pm} \Phi^{*(\bar{i},1-\bar{i})}_{\bar{j}}(\zeta)\, K^{(-)}(\zeta)_{\bar{j}}^{\bar{j}}\, \Phi^{(1-\bar{i},\bar{i})}(\zeta^{-1})$$

boson

- $T_B^{(\bar{i},-)}(\zeta)|\bar{i};-\rangle_B = |\bar{i};-\rangle_B$

# Boundary Vacuum state

$$T_B^{(i,-)}(\xi)\,|i;-\rangle_B = |i;-\rangle_B \qquad (i=0,1)$$

- $$|0;-\rangle_B = \exp_q\left(-\frac{s}{q} e_0 q^{-h_0}\right)\exp(F)\,|0\rangle$$

- $$F = \frac{1}{2}\sum_{n=1}^{\infty}\frac{-nq^{6n}}{[2n]_q[n]_q}a_{-n}^2 + \sum_{n=1}^{\infty}\left\{-\theta_n\frac{q^{\frac{5}{2}n}(1-q^n)}{[2n]_q} + \frac{q^{\frac{7}{2}n}r^n}{[2n]_q}\right\}a_{-n}$$

$$\exp_q(x) = \sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n-1)}}{[n]_q}x^n, \quad \theta_n = \begin{cases} 1 & n:\text{even} \\ 0 & n:\text{odd}\end{cases}$$

Similar for $i=1$

## Dual state

$$ {}_B\langle -;\bar{\imath}|\, T_B^{(\bar{\imath},-)}(\zeta) = {}_B\langle -;\bar{\imath}| \qquad (\bar{\imath}=0,1) $$

- ${}_B\langle -;0| = \langle 0|\exp(G)\,\exp_{q^{-1}}\left(\frac{\zeta}{q} e_0 q^{-h_0}\right)$

- $$ G = -\sum_{n=1}^{\infty} \frac{n q^{-2n}}{[2n]_q [n]_q} a_n^2 + \sum_{n=1}^{\infty} \left\{ \theta_n \frac{q^{-\frac{3}{2}n}(1-q^n)}{[2n]_q} - \frac{q^{-\frac{5}{2}n} r^n}{[2n]_q} \right\} a_n $$

[Jimbo, Kedem, Kojima, Konno, Miwa] (1995)
[Baseilhac, Belliard] (2013)
[Baseilhac, Kojima] (2014)

Similar for $\bar{\imath}=1$

# Correlation function

$$\frac{{}_B\langle \bar{i};-|E_{\varepsilon_M\varepsilon'_M}\cdots E_{\varepsilon_2\varepsilon'_2}E_{\varepsilon_1\varepsilon'_1}|-;\bar{i}\rangle_B}{{}_B\langle \bar{i};-|-;\bar{i}\rangle_B} \qquad \left(g=\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}\right)$$

$$=g^M P^{(\bar{i})}_{\varepsilon_1\cdots\varepsilon_M\varepsilon'_M\cdots\varepsilon'_1}(-q^{-1}\zeta_1,\cdots,-q^{-1}\zeta_M,\zeta_M,\cdots,\zeta_1)\Big|_{\zeta_1=\cdots=\zeta_M=1}$$

$$P^{(\bar{i})}_{\varepsilon_1\cdots\varepsilon_{2M}}(\zeta_1\zeta_2\cdots\zeta_{2M}) \qquad (\bar{i}=0,1)$$

$$=\frac{{}_B\langle \bar{i};-|\Phi^{(\bar{i},1-\bar{i})}_{\varepsilon_1}(\zeta_1)\Phi^{(1-\bar{i},\bar{i})}_{\varepsilon_2}(\zeta_2)\cdots\Phi^{(1-\bar{i},\bar{i})}_{\varepsilon_{2M}}(\zeta_{2M})|-;\bar{i}\rangle_B}{{}_B\langle \bar{i};-|-;\bar{i}\rangle_B}$$

Bosonization

## Correlation function

$A = \{ j \mid 1 \le j \le 2M,\ \varepsilon_j = + \}$

$\{z\}_\infty = (z; q^4, q^4)_\infty$

$[z]_\infty = (z; q^8, q^8)_\infty$

$M \geq |A|$

$$P^{(0)}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2M}}(\zeta_1 \zeta_2 \cdots \zeta_{2M})$$

$$= (-q^2)^{M^2 - \sum_{a \in A} a} \, (s/q^2)^{M-|A|} \, (1-q^2)^{|A|} \left( \frac{\{q^6\}_\infty}{\{q^8\}_\infty} \right)^{2M}$$

$$\times \prod_{1 \le j < k \le 2M} \frac{\{q^6 \zeta_j^2 \zeta_k^2\}_\infty \{q^2/\zeta_j^2 \zeta_k^2\}_\infty \{q^6 \zeta_j^2/\zeta_k^2\}_\infty \{q^6 \zeta_k^2/\zeta_j^2\}_\infty}{\{q^8 \zeta_j^2 \zeta_k^2\}_\infty \{q^4/\zeta_j^2 \zeta_k^2\}_\infty \{q^8 \zeta_j^2/\zeta_k^2\}_\infty \{q^8 \zeta_k^2/\zeta_j^2\}_\infty}$$

$$\times \prod_{j=1}^{2M} \frac{[q^{10} \zeta_j^4]_\infty [q^{14} \zeta_j^4]_\infty [q^{10}/\zeta_j^4]_\infty [q^6/\zeta_j^4]_\infty}{[q^{12} \zeta_j^4]_\infty [q^{16} \zeta_j^4]_\infty [q^{12}/\zeta_j^4]_\infty [q^8/\zeta_j^4]_\infty}$$

$$\times \prod_{j=1}^{2M} \zeta_j^{\frac{1+\varepsilon_j}{2} + 2M - j} \sum_{\substack{l, m \ge 0 \\ l+m = M-|A|}} \frac{(-1)^m q^{\frac{l(l-1) - m(m-1)}{2}(2i-1) - 6Mm}}{[l]_q! \, [m]_q!}$$

$$\times \oint \cdots \oint \prod_{a \in A} \frac{dw_a}{2\pi\sqrt{-1}} w_a^{1-i} \oint \cdots \oint \prod_{a=1}^{M-|A|} \frac{dv_a}{2\pi\sqrt{-1}} v_a^{1-2i} \times \cdots$$

## Correlation function

$$\times \frac{\prod_{a\in A}(w_a^2/q^2;q^4)_\infty(q^6/w_a^2;q^4)_\infty}{\prod_{a\in A}\left\{\prod_{1\le j\le a}(\xi_j^2-q^{-2}w_a)\prod_{a\le j\le 2M}(w_a-q^4\xi_j^2)\right\}}$$

$$\times \frac{\prod_{a=1}^{M-|A|}(v_a^2/q^2;q^4)_\infty(q^6/v_a^2;q^4)_\infty}{\prod_{j=1}^{2M}\left\{\prod_{a=1}^{\ell}(v_a-q^4\xi_j^2)\prod_{a=\ell+1}^{M-|A|}(\xi_j^2-q^{-2}v_a)\right\}}$$

$$\times \frac{\prod_{\substack{a,b\in A\\ a<b}} w_a^2(w_aw_b/q^2;q^2)_\infty(q^6/w_aw_b;q^2)_\infty(q^4w_a/w_b;q^2)_\infty(w_b/w_a;q^2)_\infty}{\prod_{j=1}^{2M}\prod_{a\in A}(q^2\xi_j^2w_a;q^4)_\infty(q^4/\xi_j^2w_a;q^4)_\infty(q^8\xi_j^2/w_a;q^4)_\infty(q^2w_a/\xi_j^2;q^4)_\infty}$$

$$\times \frac{\prod_{1\le a<b\le M-|A|} v_a^2(v_av_b/q^2;q^2)_\infty(q^6/v_av_b;q^2)_\infty(q^4v_a/v_b;q^2)_\infty(v_b/v_a;q^2)_\infty}{\prod_{j=1}^{2M}\prod_{a=1}^{M-|A|}(q^2\xi_j^2v_a;q^4)_\infty(q^4/v_j\xi_j^2;q^4)_\infty(q^8\xi_j^2/v_a;q^4)_\infty(q^2v_a/\xi_j^2;q^4)_\infty}$$

$\times$ ...

# Correlation function

$$\times \prod_{a=1}^{\ell} \prod_{b \in A} v_a^2 (v_a w_b/q^2; q^2)_\infty (q^4 v_a/w_b; q^2)_\infty (w_b/v_a; q^2)_\infty (q^6/v_a w_b; q^2)_\infty$$

$$\times \prod_{a=\ell+1}^{M-A} \prod_{b \in A} w_b^2 (v_a w_b/q^2; q^2)_\infty (v_a/w_b; q^2)_\infty (q^4 w_b/v_a; q^2)_\infty (q^6/v_a w_b; q^2)_\infty$$

$$\times \prod_{j=1}^{2M} \frac{(q^2 r \zeta_j^2; q^4)_\infty}{(q^4 r \zeta_j^2; q^4)_\infty} \frac{1}{\prod_{a \in A} (1 - r w_a/q^2) \prod_{a=1}^{M-|A|} (1 - q^2/r v_a)}$$

$\boxed{\text{black}}$ the same as diagonal

$\boxed{\text{red}}$ new

[Baseilhac, Kojima] (2014)

## Correlation function

$$\frac{{}_B\langle 0;- | \sigma_1^z | -;0\rangle_B}{{}_B\langle 0;- | -;0\rangle_B} = -1-2(1-r)^2 \sum_{n=1}^{\infty} \frac{(-q^2)^n}{(1-rq^{2n})^2}$$

$$\frac{{}_B\langle 0;- | \sigma_1^+ | -;0\rangle_B}{{}_B\langle 0;- | -;0\rangle_B} = S\left(2+(1-r)\sum_{n=1}^{\infty}(-q^2)^n \frac{2q^{2n}-r(1+q^{4n})}{(1-rq^{2n})^2}\right)$$

$$\frac{{}_B\langle 0;- | \sigma_1^- | -;0\rangle_B}{{}_B\langle 0;- | -;0\rangle_B} = 0$$

[Baseilhac, Kojima] (2014)

## Correlation function

$$\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_M = \varepsilon_1'+\varepsilon_2'+\cdots+\varepsilon_M' \Longrightarrow$$

$$\frac{{}_B\langle \bar{i};\pm| E_{\varepsilon_M\varepsilon_M'} \cdots E_{\varepsilon_2\varepsilon_2'} E_{\varepsilon_1\varepsilon_1'} |\pm;\bar{i}\rangle_B}{{}_B\langle \bar{i};\pm|\pm;\bar{i}\rangle_B} \quad \text{Triangular}$$

$$= \frac{{}_B\langle \bar{i}| E_{\varepsilon_M\varepsilon_M'} \cdots E_{\varepsilon_2\varepsilon_2'} E_{\varepsilon_1\varepsilon_1'} |\bar{i}\rangle_B}{{}_B\langle \bar{i}|\bar{i}\rangle_B} \quad \text{Diagonal}$$

Ex.

$$\frac{{}_B\langle 0;-|\sigma_1^z|-;0\rangle_B}{{}_B\langle 0;-|-;0\rangle_B} = \frac{{}_B\langle 0|\sigma_1^z|0\rangle_B}{{}_B\langle 0|0\rangle_B}$$

# $\boxed{\text{Spin-Reversal Property}}$

Identities between
Correlation Functions

# Spin-Reversal Property

$$\text{Upper} \quad \begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \qquad \sigma^+ = \begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad \sigma^z = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad (r, s)$$

$$\updownarrow$$

$$\text{Lower} \quad \begin{pmatrix}0\\1\end{pmatrix} \quad \begin{pmatrix}1\\0\end{pmatrix} \qquad \sigma^- = \begin{pmatrix}0&0\\1&0\end{pmatrix} \qquad -\sigma^z = \begin{pmatrix}-1&0\\0&1\end{pmatrix} \qquad (1/r, -s/r)$$

$$\boxed{\begin{array}{ll} \text{Upper} & H_B^{(+)} = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\sigma_{n+1}^x\sigma_n^x + \sigma_{n+1}^y\sigma_n^y + \Delta\sigma_{n+1}^z\sigma_n^z\right) - \frac{1-q^2}{4q}\frac{1+r}{1-r}\sigma_1^z - \frac{s}{1-r}\sigma_1^+ \\ & \updownarrow \\ \text{Lower} & H_B^{(-)} = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\sigma_{n+1}^x\sigma_n^x + \sigma_{n+1}^y\sigma_n^y + \Delta\sigma_{n+1}^z\sigma_n^z\right) - \frac{1-q^2}{4q}\frac{1+r}{1-r}\sigma_1^z - \frac{s}{1-r}\sigma_1^- \end{array}}$$

# Spin-Reversal Property

$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leftrightarrow \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Upper $\quad u_\pm \longleftrightarrow u_\mp \quad$ Lower

$$\boxed{\left.\frac{{}_B\langle 0;+|\,\Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_{2M}}(\zeta_{2M})\,|+;0\rangle_B}{{}_B\langle 0;+|+;0\rangle_B}\right|_{\substack{r\to 1/r \\ s\to -s/r}} = \frac{{}_B\langle 1;-|\,\Phi_{-\varepsilon_1}(\zeta_1)\cdots\Phi_{-\varepsilon_{2M}}(\zeta_{2M})\,|-;1\rangle_B}{{}_B\langle 1;-|-;1\rangle_B}}$$

$$(r, s) \longleftrightarrow (1/r, -s/r)$$

## Example

$$\boxed{\begin{aligned} &\frac{{}_B\langle +;1|\, \Phi_+^*(-q^{-1}\xi)\, \Phi_+(\xi)\, |1;+\rangle_B}{{}_B\langle +;1|1;+\rangle_B} \\ &= \left.\frac{{}_B\langle -;0|\, \Phi_-^*(-q^{-1}\xi)\, \Phi_-(\xi)\, |0;-\rangle_B}{{}_B\langle -;0|0;-\rangle_B}\right|_{\substack{r \mapsto 1/r \\ s \mapsto -s/r}} \end{aligned}}$$

## Spin-Reversal Property

$$2+\frac{1-z/r}{z}\sum_{k=1}^{\infty}(-q^2)^k\frac{(z-z^{-1})+(1+q^{4k})/r+(z+z^{-1})q^{2k}}{(1-q^{2k}z/r)(1-q^2/rz)}$$

$$=\frac{(q^2;q^2)_\infty^4}{(q^4;q^4)_\infty^2}\left(\iiint_{C_1}-q^{-2}\iiint_{C_2}\right)\prod_{a=1}^{3}\frac{dw_a}{2\pi\sqrt{-1}}\,\frac{w_1}{w_3^2}$$

$$\times\frac{(1-q^2)(w_3-q/r)}{\prod_{a=1}^{2}(w_a-q^2r)\,(w_2-q^2w_1)(w_1w_2-q^4)}$$

$$\times\frac{\Theta_{q^2}(w_1w_2/q^2)\,\Theta_{q^2}(q^2w_1/w_2)\,\Theta_{q^2}(zw_3/q^2)\,\Theta_{q^2}(qw_3/z)\prod_{a=1}^{3}\Theta_{q^4}(w_a^2/q^2)}{\prod_{a=1}^{2}\Theta_{q^2}(w_aw_3/q^2)\,\Theta_{q^2}(w_a/qw_3)\,\Theta_{q^2}(w_az)\,\Theta_{q^2}(q^2w_a/z)}$$

## $\boxed{\text{Example}}$

## Form factor

$$
\boxed{
\begin{aligned}
&{}_B\langle +;0|\,\Psi_+^*(\xi_1)\cdots\Psi_+^*(\xi_M)\,\Phi_+(\zeta_1)\cdots\Phi_+(\zeta_M)\,|0;+\rangle_B \\
&= {}_B\langle -;1|\,\Psi_-^*(\xi_1)\cdots\Psi_-^*(\xi_M)\,\Phi_-(\zeta_1)\cdots\Phi_-(\zeta_M)\,|1;-\rangle_B \Big|_{\substack{r\to 1/r \\ \zeta\to -\zeta/r}}
\end{aligned}
}
$$

# Spin-Reversal Property

$$\prod_{j=1}^{M} \frac{(q^4 r \xi_j^2 ; q^4)_\infty (r/\xi_j^2 ; q^4)_\infty (q r \xi_j^2 ; q^4)_\infty (q r/\xi_j^2 ; q^4)_\infty}{(q^2 r \xi_j^2 ; q^4)_\infty (q^2 r/\xi_j^2 ; q^4)_\infty (q^3 r \xi_j^2 ; q^4)_\infty (r/q\xi_j^2 ; q^4)_\infty}$$

$$= q^M (1-q^2)^M q^{-3M(M+1)} (q^2 ; q^2)_\infty^{4M} (q^4 ; q^4)_\infty^{-4M^2-2M}$$

$$\times \oint \cdots \oint_C \prod_{a=1}^{M} \frac{dw_a}{2\pi\sqrt{-1}} w_a \prod_{b=1}^{M} \frac{du_b}{2\pi\sqrt{-1}}$$

$$\times \prod_{a=1}^{M} \frac{(1-r u_a/q^3)}{(1-r w_a/q^2)} \times \prod_{1\le a<b\le M} \frac{(1-q^4/u_a u_b)(1-q^2 u_a/u_b)}{(1-q^4/w_a w_b)(1-q^2 w_a/w_b)} \times$$

$$\times \prod_{a=1}^{M} \frac{\Theta_{q^4}(w_a^2/q^2)\,\Theta_{q^4}(u_a^2/q^2)}{\prod_{b=1}^{M} w_a^2\, \Theta_{q^2}(w_a u_b/q^3)\,\Theta_{q^2}(u_b/q w_a)} \times \cdots$$

....

$$\times \frac{\prod_{1\le a<b\le M} W_a^2\, \theta_{q^2}(W_aW_b/q^2)\, \theta_{q^2}(W_b/W_a)}{\prod_{a=1}^{M}\left\{\prod_{1\le \bar{j}\le a}(\xi_{\bar{j}}^2 - q^{-2}W_a)\prod_{a\le \bar{j}\le M}(W_a - q^4\xi_{\bar{j}}^2)\right\}}$$

$$\times \frac{\prod_{1\le a<b\le M} u_a^2\, \theta_{q^2}(u_au_b/q^4)\, \theta_{q^2}(u_b/q^2u_a)}{\prod_{a=1}^{M}\left\{\prod_{1\le \bar{j}\le a}(\xi_{\bar{j}}^2 - q^{-4}u_a)\prod_{a\le \bar{j}\le M}(u_a - q^2\xi_{\bar{j}}^2)\right\}}$$

$$\times \prod_{\bar{j}=1}^{M}\frac{\prod_{b=1}^{M} u_b\, \theta_{q^4}(q\xi_{\bar{j}}^2u_b)\, \theta_{q^4}(q^3\xi_{\bar{j}}^2/u_b)}{\prod_{a=1}^{M}(q^2\xi_{\bar{j}}^2W_a;q^4)_\infty(q^8\xi_{\bar{j}}^2/W_a;q^4)_\infty(q^2W_a/\xi_{\bar{j}}^2;q^4)_\infty(q^4/\xi_{\bar{j}}^2W_a;q^4)_\infty}$$

$$\times \prod_{\bar{j}=1}^{M}\frac{\prod_{a=1}^{M} W_a\, \theta_{q^4}(q\xi_{\bar{j}}^2W_a)\, \theta_{q^4}(q^3\xi_{\bar{j}}^2/W_a)}{\prod_{b=1}^{M}(\xi_{\bar{j}}^2u_b;q^4)_\infty(q^6\xi_{\bar{j}}^2/u_b;q^4)_\infty(u_b/\xi_{\bar{j}}^2;q^4)_\infty(q^2/\xi_{\bar{j}}^2u_b;q^4)_\infty}$$

## Example

ABF model

Spin-Reversal $(c, k, \pm) \longleftrightarrow (-c, r-k, \mp)$

$$\boxed{\frac{{}_B\langle k,k \,|\, \Phi_-(z_1)\,\Phi_+(z_2) \,|\, k,k\rangle_B}{{}_B\langle k,k \,|\, k,k\rangle_B} = \frac{{}_B\langle k-1,k \,|\, \Phi_+(z_1)\,\Phi_-(z_2) \,|\, k-1,k\rangle_B}{{}_B\langle k-1,k \,|\, k-1,k\rangle_B}\Bigg|_{\substack{c \mapsto -c \\ k \mapsto r-k}}}$$

We checked this for $r \to \infty$.

# Spin-Reversal

$$\prod_{j=1}^{2} z_j^{\frac{k}{2r}} \frac{(x^{2c+2k+2} z_j ; x^4, x^{2r})_\infty (x^{2r-2c+2} z_j ; x^4, x^{2r})_\infty (x^{2c}/z_j ; x^4, x^{2r})_\infty (x^{2r-2c-2k}/z_j ; x^4, x^{2r})_\infty}{(x^{2c+2k+4} z_j ; x^4, x^{2r})_\infty (x^{2r-2c+4} z_j ; x^4, x^{2r})_\infty (x^{2c+2}/z_j ; x^4, x^{2r})_\infty (x^{2r-2c-2k+2}/z_j ; x^4, x^{2r})_\infty}$$

$$\times \oint_C \frac{dw}{2\pi\sqrt{-1}} w^{-\frac{k}{r}-1} \frac{[u - v_2 + \frac{1}{2} + k]}{[u - v_2 + \frac{1}{2}]} \times \frac{(x^6 w^2 ; x^4, x^{2r})_\infty (x^2/w^2 ; x^4, x^{2r})_\infty}{(x^{2r+4} w^2 ; x^4, x^{2r})_\infty (x^{2r}/w^2 ; x^4, x^{2r})_\infty}$$

$$\times \prod_{j=1}^{2} \frac{(x^{2r+3} z_j w ; x^4, x^{2r})_\infty (x^{2r-1}/z_j w ; x^4, x^{2r})_\infty}{(x^5 z_j w ; x^4, x^{2r})_\infty (x/z_j w ; x^4, x^{2r})_\infty}$$

$$\times \prod_{j=1}^{2} z_j^{-\frac{r-1}{r}} \frac{(x^{2r+3} z_j/w ; x^4, x^{2r})_\infty (x^{2r-1} w/z_j ; x^4, x^{2r})_\infty}{(x^5 z_j/w ; x^4, x^{2r})_\infty (x w/z_j ; x^4, x^{2r})_\infty}$$

$$\times \frac{1}{(x^{2c+2k+1} w ; x^{2r})_\infty (x^{2r-2c+1} w ; x^{2r})_\infty (x^{2c-1}/w ; x^{2r})_\infty (x^{2r-2c-2k-1}/w ; x^{2r})_\infty}$$

$= \cdots\cdots$

$$
\cdots = \prod_{j=1}^{2} z_j^{-\frac{k}{2r}} \frac{(x^{2r-2c-2k+2} z_j ; x^4, x^{2r})_\infty (x^{2c+2} z_j ; x^4, x^{2r})_\infty (x^{2r-2c}/z_j ; x^4, x^{2r})_\infty (x^{2c+2k}/z_j ; x^4, x^{2r})_\infty}{(x^{2r-2c-2k+4} z_j ; x^4, x^{2r})_\infty (x^{2c+4} z_j ; x^4, x^{2r})_\infty (x^{2r-2c+2}/z_j ; x^4, x^{2r})_\infty (x^{2c+2k+2}/z_j ; x^4, x^{2r})_\infty}
$$

$$
\times \oint_C \frac{dw}{2\pi\sqrt{-1}} w^{\frac{k}{r}-1} \frac{[u - v_1 + \frac{1}{2} - k - 1]}{[u - v_1 + \frac{1}{2}]} \times \frac{(x^6 w^2 ; x^4, x^{2r})_\infty (x^2/w^2 ; x^4, x^{2r})_\infty}{(x^{2r+2} w^2 ; x^4, x^{2r})_\infty (x^{2r}/w^2 ; x^4, x^{2r})_\infty}
$$

$$
\times \prod_{j=1}^{2} \frac{(x^{2r+3} z_j w ; x^4, x^2)_\infty (x^{2r-1}/z_j w ; x^4, x^{2r})_\infty}{(x^5 z_j w ; x^4, x^{2r})_\infty (x/z_j w ; x^4, x^{2r})_\infty}
$$

$$
\times w^{-\frac{r-1}{r}} \frac{(x^{2r-1} z_2/w ; x^4, x^{2r})_\infty (x^{2r+3} w/z_2 ; x^4, x^{2r})_\infty}{(x z_2/w ; x^4, x^{2r})_\infty (x^5 w/z_2 ; x^4, x^{2r})_\infty}
$$

$$
\times z_1^{-\frac{r-1}{r}} \frac{(x^{2r+3} z_1/w ; x^4, x^{2r})_\infty (x^{2r-1} w/z_1 ; x^4, x^{2r})_\infty}{(x^5 z_1/w ; x^4, x^{2r})_\infty (x w/z_1 ; x^4, x^{2r})_\infty}
$$

$$
\times \frac{1}{(x^{2r-2c-2k+1} w ; x^{2r})_\infty (x^{2c+1} w ; x^{2r})_\infty (x^{2r-2c-1}/w ; x^{2r})_\infty (x^{2c+2k-1}/w ; x^{2r})_\infty}
$$

# Spin-Reversal Property

Triangular $U_q(\widehat{sl}_2) \rightsquigarrow U_{qp}^{\text{Open}}$

$$2+\frac{1-z/r}{z}\sum_{k=1}^{\infty}(-q^2)^k\frac{(z-z^{-1})+(1+q^{4k})/r+(z+z^{-1})q^{2k}}{(1-q^{2k}z/r)(1-q^2/rz)}$$

$$=\frac{(q^2;q^2)_\infty^4}{(q^4;q^4)_\infty^2}\left(\iiint_{C_1}-q^{-2}\iiint_{C_2}\right)\prod_{a=1}^{3}\frac{dw_a}{2\pi\sqrt{-1}}\,\frac{w_1}{w_3^2}$$

$$\times\frac{(1-q^2)(w_3-q/r)}{\prod_{a=1}^{2}(w_a-q^2r)\,(w_2-q^2w_1)(w_1w_2-q^4)}$$

$$\times\frac{\Theta_{q^2}(w_1w_2/q^2)\,\Theta_{q^2}(q^2w_1/w_2)\,\Theta_{q^2}(zw_3/q^2)\,\Theta_{q^2}(qw_3/z)\,\prod_{a=1}^{3}\Theta_{q^4}(w_a^2/q^2)}{\prod_{a=1}^{2}\Theta_{q^2}(w_aw_3/q^2)\,\Theta_{q^2}(w_a/qw_3)\,\Theta_{q^2}(w_az)\,\Theta_{q^2}(q^2w_a/z)}$$

## Solved Models

Vertex Operator Approach

- Diagonal

| | | |
|---|---|---|
| Quantum groups | $U_q(\widehat{sl_2})$ | [Jimbo, Kedem, Kojima, Konno, Miwa] (1995) |
| | $U_q(\widehat{sl_N})$ | [Furutsu, Kojima] (2000) |
| | $U_q(A_2^{(2)})$ | [Kojima] (2011) |
| | $U_q(\widehat{sl}(M\|N))$ | [Kojima] (2013) |
| Elliptic algebra | $U_{q,p}(\widehat{sl_2})$ | [Miwa, Weston] (1998) |
| | $U_{q,p}(\widehat{sl_N})$ | [Kojima] (2011) |

- Non-Diagonal

Triangular $U_q(\widehat{sl_2})$ [Baseilhac, Kojima] (2014)

# Summary

- We studied Boundary $sl_N$-ABF model.
  We constructed integral representations of correlation functions.
- We studied Triangular Boundary XXZ chain.
  We constructed integral representations of correlation functions.
- We conjectured identities of integrals of theta function using spin-reversal property of the above models.

# References

[Jimbo, Kedem, Kojima, Konno, Miwa] Nucl. Phys. B441 (1995)

[Furutsu, Kojima] JMP 41 (2000)

[Kojima, Konno] CMP 239 (2003)

[Miwa, Weston] Nucl. Phys. B486 (1997)

[Kojima] JMP 52 (2011)

[Baseilhac, Beillard] Nucl. Phys. B873 (2013)

[Baseilhac, Kojima] Nucl. Phys. B880 (2014)

[Baseilhac, Kojima] to appear in J. Stat. Mech.
arXiv: 1404.049