# Another modification of the ACD sum rule

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Outline
1. Introduction: Analytic Continuation by Duality (ACD)
2. ACD and Modified ACD
3. Approximation Routine for Modified ACD
4. Derivation of the ACD Estimate
5. Model Spectra
6. Conclusions

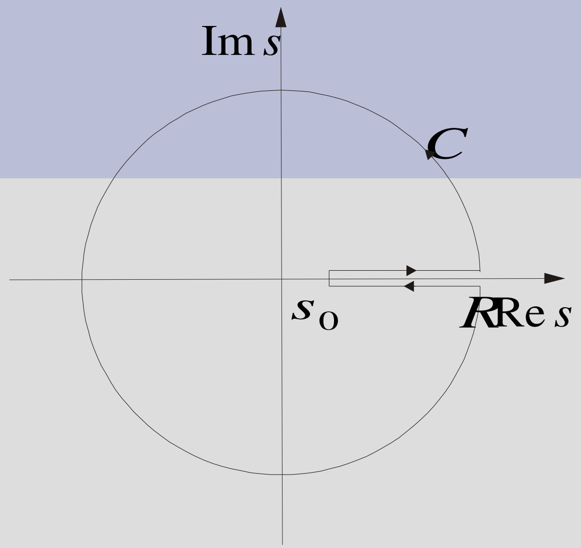
# Introduction: Analytic Continuation by Duality (ACD)

- Finite-energy QCD sum rule
- Usually, subasymptotic QCD (Shifman-Vainshtein-Zakharov)
- Low-energy phenomenology
  - N. F. Nasrallah, K. Schilcher et al. (1982-)
  - critique: Caprini & Verzegnassi (1984)
  - Liu Jueping et al. (1992-)
- Application beyond Standard Model:
  - Sundrum & Hsu (1993) – Technicolor
  - critique: Ignjatović, Takeuchi, Wijewardhana (1997-)
- Modifications of the ACD:
  - Nasrallah (2001) – yields only a lower bound; abandoned
  - Dominguez, Nasrallah et al. (2007-)
  - Li Minghai & Liu Jueping (2012) – subject of this talk **Reliability and stability checked for simple model spectra**

# 2. ACD and Modified ACD

Cauchy's integral formula:

$$F(t)=\frac{1}{2\pi i}\oint_C \frac{F(s)}{s-t}\,ds.$$




*F(t)* – any function analytical in *C*

Discontinuity relation:

$$F(s+i\varepsilon)-F(s-i\varepsilon)=2i\,\mathrm{Im}\,F(s+i\varepsilon)$$

Integral is divided into two parts

$$F(t)=\frac{1}{2\pi i}\oint_{|s|=R} \frac{F(s)}{s-t}\,ds+\frac{1}{\pi}\int_{s_0}^{R}\frac{\mathrm{Im}\,F(s)}{s-t-i\varepsilon}\,ds.$$

*F(t)* can be simply related to vacuum polarization $\Pi(t)$ – the case in ACD, or we can take $F(t) = \Pi(t)\,\rho(t)$, $\rho(t)$ being a **real** weight function and $\rho(0)=1$ – the case in **modified ACD**

$$\Pi(0)=\frac{1}{2\pi i}\oint_{|s|=R} \frac{\Pi(s)\rho(s)}{s}\,ds+\frac{1}{\pi}\int_{s_0}^{R}\frac{\mathrm{Im}\,\Pi(s)\rho(s)}{s}\,ds.$$

**ACD makes the second integral vanish!**

# 3. Approximation Routine for Modified ACD

- Minimization of the second integral is achieved through best-$L_2$ norm weighted by the function $\rho(s) = D_1(s) D_2(s)$
- The problem is technically involved, but essentially simple (though finding best-$L_1$ or $L_\infty$ approximations is highly non-trivial)
- Since the (usually asymptotic!) expression for $\Pi(s)$ tends to be better near the upper bound $R$ it makes sense to suppose

$$\frac{d^k}{ds^k} D_2(s)\Big|_{s=R} = 0, \quad k = 0, \ldots, N_2 \;\rightarrow\; D_2(s) = \left(1 - \frac{s}{R}\right)^{N_2+1}$$

This was the idea of the 2001 modification, but it gives only a lower bound if there is no (nontrivial) $D_1(s)$

We minimize

$$||d|| = \left[\int_{s_0}^{R} |d(s)|^2 D_2(s)\, ds\right]^{\frac{1}{2}},$$

where

$$d(s) = \frac{D_1(s)}{s} = \frac{1}{s} - p_{N_1}(s) = \frac{1}{s} - \sum_{n=0}^{N_1} a_n(N_1, N_2) s^n ¿$$

On change of variables $x=-ps+q$, where

$$p=\frac{2}{R-s_o}, \quad q=\frac{R+s_o}{R-s_o}$$

reduces the approximation problem to minimization of

$$\|d\|=\left[\int_{-1}^{1}\left(\frac{1}{x-q}-\sum_{n=0}^{N_1} b_n(N_1,N_2)x^n\right)^2(1+x)^{N_2+1}dx\right]^{\frac{1}{2}}.$$

It is known that the polynomials which minimize this norm are Jacobi polynomials

$$y_n(x)=P_n^{(0,N_2+1)}(x), \quad n=0,\ldots,N_1.$$

Of course, Jacobi polynomials are orthogonal with the above weight:

$$\int_{-1}^{1} y_m(x)\,y_n(x)(1+x)^{N_2+1}dx=d_n^2\delta_{mn}$$

where $d_n=d_n(N_2)$ is a constant. Hence we fit the function $1/(x-q)$ to the truncated Fourier-Jacobi series

$$\frac{1}{x-q}\approx\sum_{n=0}^{N_1} c_n y_n(x), \quad c_n=\frac{1}{d_n^2}\int_{-1}^{1}\frac{y_n(x)(1+x)^{N_2+1}}{x-q}dx$$

The Fourier-Jacobi coefficients can be expressed through the Jacobi functions of the second kind:

$$c_n(q) = -2\frac{Q_n(q)}{d_n^2}(1+q)^{N_1}$$

Jacobi functions of the second kind are computable analitically because they obey a recursive relation; namely, they are corecursive with the Jacobi polynomials.

Eventually we find the coefficients of the polynomial

$$\rho(s) = D_1(s)D_2(s) = \sum_{n=0}^{N_1+N_2+2} A_n s^n$$

which enables us to derive the ACD estimate.

# 4. Derivation of the ACD Estimate

Returning to the main ACD formula we find the approximation

$$\Pi(0)=\frac{1}{2\pi i}\oint_{|s|=R}\frac{\Pi(s)\rho(s)}{s}ds.+\Delta_{\text{fit}}$$

where $\Delta_{\text{fit}}$ is the fit error. Usually, the vacuum polarization is found from the Operator Product Expansion (OPE) derived by Shifman, Vainshtein and Zakharov for subasymptotic QCD:

$$\Pi(s)=\sum_{m=1}^{M}\frac{h_m(s)}{s^m}+O\left(\frac{1}{s^{M+1}}\right),$$

which is an asymptotic series, so only a few terms are known. This introduces truncation error $\Delta_{\text{trunc}}$. In ACD, it is often assumed that the OPE (Wilson) coefficients do not depend on the momentum, but this assumption is *not* necessary. Neglecting the *s*-dependence we find

$$\Pi_{\text{ACD}}=\sum_{n=0}^{\min(N_1+N_2+2,M-1)}\hat{h}_{n+1}A_n(N_1,N_2),$$

although this introduces another error, $\Delta_{\text{AC}}$.

# 5. Model Spectra

To investigate reliability and stability of the ACD-derived estimates, we use model spectra. These are simple cases in which the sepctrum is known exactly so we can compare ACD estimates to the exact values.

- ACD is an ill-posed problem so difficulties can be expected.
- Oscillatory behavior was previously seen, probably due to the oscillatory nature of the best-$L_p$ approximations

a) Vector-meson dominance model with infinitely sharp resonances

$$F(s)=\frac{f_V^2}{s-m_V^2+i\varepsilon}-\frac{f_A^2}{s-m_A^2+i\varepsilon}, \qquad F(0)=\frac{f_A^2}{m_A^2}-\frac{f_V^2}{m_V^2};$$

$$\mathrm{Im}\,F(s)=-\pi[f_V^2\,\delta(s-m_V^2)-f_A^2\,\delta(s-m_A^2)],$$

b) Vector-meson dominance model with finite-width resonances

$$F(s)=\frac{f_V^2}{s-m_V^2+i\sqrt{s}\,\Gamma_V}-\frac{f_A^2}{s-m_A^2+i\sqrt{s}\,\Gamma_A}, \quad F=\frac{f_A^2}{m_A^2}-\frac{f_V^2}{m_V^2},$$

$$\mathrm{Im}\,F(s)=-\sqrt{s}\;\Theta(s-s_0)\left[\frac{f_V^2\,\Gamma_V}{(s-m_V^2)^2+s\Gamma_V^2}-\frac{f_A^2\,\Gamma_A}{(s-m_A^2)^2+s\Gamma_A^2}\right].$$

The parameters of the models are fixed by Weinberg sum rules:

$$f_V^2 - f_A^2 = f^2, \qquad f_V^2 m_V^2 - f_A^2 m_A^2 = 0,$$

where $f$ is a constant (in low-energy QCD – pion decay constant)

Introducing parameter

$$r = \frac{m_V^2}{m_A^2}$$

(in low-energy QCD – $r = 0.4$)

$$F(0) = (1+r)\frac{f^2}{m_V^2}$$

The large-momentum expansion (for inifinitely sharp resonances)

$$F(s) = \sum_{n=0}^{\infty} \frac{f_V^2 m_V^{2\mathrm{n}} - f_A^2 m_A^{2\mathrm{n}}}{s^{n+1}} = \frac{f^2}{s} + \sum_{n=2}^{\infty} \frac{f^2 m_V^{2\mathrm{n}}}{s^{n+1}} \frac{1 - r^{1-\frac{n}{2}}}{1-r}$$

Thus

$$h_1 = f^2, \qquad h_{n+1} = f^2 m_V^{2\mathrm{n}} \frac{1 - r^{1-\frac{n}{2}}}{1-r} \qquad (n > 1)$$

The ACD-derived estimates can be compared to the exact value. It takes several terms for ACD estimates to approach the exact value. In realistic applications we do not have that many terms in the OPE.

# 6. Conclusions

- ACD is a finite-energy QCD sum rule based on OPE for subasymptotic QCD (Nasrallah and Schilcher, 1982)
- ACD is mainly used for low-energy phenomenology, although it has been used in technicolor as well
- 2001 ”simpified” ACD gives only a lower bound for estimates
- 2012 ”modified” ACD is a major departure from the original ACD
- Problems with the ACD:
  - Ill-posed problem
  - Oscillatory behavior of the estimates
  - Strong dependence on the lower and upper bound (the latter one cannot be fixed from physics)
  - Modified ACD introduces another free parameter: there are at least 3 free parameters, although they are ’somewhat’ constrained
- **ACD is not sufficiently reliable to be the method of choice for non-perturbative QCD**
- In some cases its estimates can be used for comparison purposes (e. g. with methods not based on subasymptotic QCD).