p-ADIC MATHEMATICAL PHYSICS AND ITS APPLICATIONS

Branko Dragovich
http://www.ipb.ac.rs/˜dragovich
dragovich@ipb.ac.rs

Institute of Physics, University of Belgrade
Belgrade, Serbia

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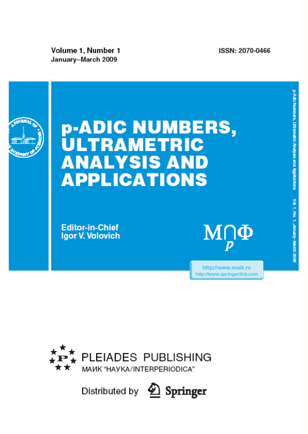
1. Introduction: *p*-adic mathematical physics

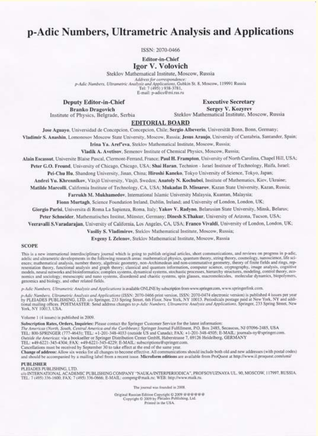
- This talk is related to *p*-adic numbers, adeles, their analysis and applications
- Application of *p*-adic numbers in modeling some physical systems started in 1987 by construction of *p*-adic scattering string amplitudes (Volovich, ...)
- *p*-Adic mathematical physics

1. Introduction: applications in various parts of modern mathematical physics and related fields

- *p*-adic and adelic string theory
- *p*-adic and adelic quantum mechanics and (quantum) field theory
- *p*-adic and adelic gravity and (quantum) cosmology
- *p*-adic and adelic space-time structure at the Planck scale
- *p*-adic and adelic dynamical systems
- *p*-adic stochastic processes
- *p*-adic aspects of information theory
- *p*-adic structure of the genetic code and protein dynamics
- *p*-adic wavelets
- spin glasses, *p*-adic unordered systems, ...

1. Introduction: international interdisciplinary journal on *p*-adics





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Address for correspondence:
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E-mail: p-adics@mi.ras.ru

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1. Introduction: international conferences on *p*-adic mathematical physics

2nd International Conference on p-Adic Mathematical Physics
Belgrade, 2005

2. p-Adic numbers, adeles and their functions

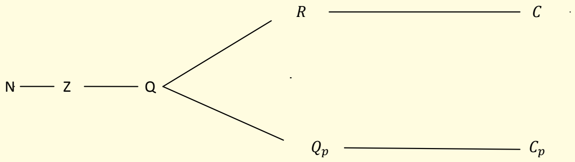
- p-Adic numbers discovered by Kurt Hensel (1861-1941) in 1897.
- Any p-adic number ($x \in \mathbb{Q}_p$) has a unique canonical representation, (p = 2,3,5,7,11, ... = a prime number)

$$x = p^{\nu} \sum_{n=0}^{+\infty} x_n p^n \,, \;\; \nu \in \mathbb{Z} \,, \;\; x_n \in \{0, \, 1, \, \cdots \, , \, p-1\}$$

$$-1 = 2 + 2 \cdot 3 + 2 \cdot 3^2 + ... + 2 \cdot 3^n + ... \quad (p = 3)$$

2. p-Adic numbers, adeles and their functions

- Ostrowski theorem: Any nontrivial norm on $\mathbb{Q}$ is equivalent to usual absolute value or to p-adic norm, where p is any prime number ($p = 2, 3, 5, 7, 11, \dots$) .
- p-adic norm of $x \in \mathbb{Q}$: $\quad |x|_p = |p^\nu \frac{a}{b}|_p = p^{-\nu}\,, \quad \nu \in \mathbb{Z}$.
- Completion of $\mathbb{Q}$ with respect to p-adic distance gives the field $\mathbb{Q}_p$ of p-adic numbers, in analogous way to construction of the field $\mathbb{R}$ of real numbers.
- $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ and $\mathbb{R}$.
- $\mathbb{Q}_p$ is totally disconnected space.



2. p-Adic numbers, adeles and their functions: some topological properties of p-adic numbers

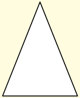
- p-adic distances $d_p(x, y) = |x - y|_p$ are non-Archimedean (ultrametric): $d_p(x, y) \leq max\{d_p(x, z), d_p(z, y)\}$
- open ball of radius r and centre a:

$$B_a(r^-) = \{x \in \mathbb{Q}_p \; : |x - a|_p < r\}$$

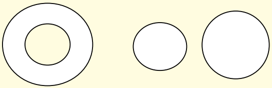
- closed ball of radius r and centre a:

$$B_a(r) = \{x \in \mathbb{Q}_p \; : |x - a|_p \leq r\}$$

- balls are simultaneously closed and open



All triangles are isosceles.



There is no partial intersection of balls.

Any point of a ball is its center.

- There are mainly two kinds of analysis of p-adic variable:
  (i) *p-adic valued functions of p-adic variable*
  (ii) complex (real) valued functions of p-adic variable.

- *Results of measurements are (rational) subset of real numbers – p-adic number cannot be directly measured*

- *Analysis of complex (real) valued functions of p-adic (and real) variables is necessary to connect models with measurements.*

2. *p*-Adic numbers, adeles and their functions

- Real and *p*-adic numbers are unified by adeles. An adele α is an infinite sequence

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \cdots, \alpha_p, \cdots), \quad \alpha_\infty \in \mathbb{R}, \ \alpha_p \in \mathbb{Q}_p$$

where for all but a finite set $\mathcal{P}$ of primes *p* one has that $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \le 1\}$, i.e. *p*-adic integers.

- Space of adeles

$$\mathbb{A} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p .$$

C. Chevalley (1909 - 1984)

A. Weil (1906 - 1998)

2. p-Adic numbers, adeles and their functions

Some connections of p-adic and real norms of rational numbers

$$|x|_\infty^c \times \prod_{p\in\mathbb{P}} |x|_p^c = 1\,, \;\; \text{if } x \in \mathbb{Q}^\times$$

$$\chi_\infty(x) \times \prod_{p\in\mathbb{P}} \chi_p(x) = 1\,, \;\; \text{if } x \in \mathbb{Q}$$

$$\chi_\infty(x) = \exp(-2\pi i x), \qquad \chi_p(x) = \exp 2\pi i \{x\}_p$$

$$\prod_{p\in\mathbb{P}} \Omega(|x|_p) = \begin{cases} 1, & x \in \mathbb{Z}, \\ 0, & x \in \mathbb{Q} \setminus \mathbb{Z} \end{cases}$$

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \le 1, \\ 0, & |x|_p > 1 \end{cases}$$

2. p-Adic numbers, adeles and their functions

- Numerical result of any measurement is a (real) rational number with some uncertainty (error): i.e. $x = \bar{x} \pm \Delta x$
- Numerical result of the corresponding theoretical model has to be equal to this rational number from the measurement.
- Note that p-adic number is not result of a measurement!
- Then the question: How p-adic numbers and p-adic analysis can be related to applications in physics and sciences?

2. *p*-Adic numbers, adeles and their functions

- Recall standard theoretical models

$$\textit{classical models}: \ \mathbb{R} \to \mathbb{R}$$

$$\textit{quantum models}: \ \mathbb{R} \to \mathbb{C} \to \mathbb{R}$$

- *p*-Adic applications

$$\textit{classical models}(\textit{genetic code}, \ldots): \ \mathbb{Q}_p \to \mathbb{R}$$

$$\textit{quantum models}(\textit{string theory}, \ldots): \mathbb{Q}_p \to \mathbb{C} \to \mathbb{R}$$

- *p*-Adic side of some phenomena emerges at a deeper level.

3. p-Adic strings

Volovich, Freund, Witten, Vladimirov, Arefeva, B. D., ...

String amplitudes (1987):

- standard crossing symmetric Veneziano amplitude

$$A_\infty(a,b) = g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1} \, |1-x|_\infty^{b-1} \, d_\infty x$$
$$= g_\infty^2 \, \frac{\zeta(1-a)}{\zeta(a)} \, \frac{\zeta(1-b)}{\zeta(b)} \, \frac{\zeta(1-c)}{\zeta(c)}$$

- p-adic crossing symmetric Veneziano amplitude

$$A_p(a,b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1} \, |1-x|_p^{b-1} \, d_p x$$
$$= g_p^2 \, \frac{1-p^{a-1}}{1-p^{-a}} \, \frac{1-p^{b-1}}{1-p^{-b}} \, \frac{1-p^{c-1}}{1-p^{-c}}$$

where $a = -s/2 - 1$ and $a, b, c \in \mathbb{C}$ and $a + b + c = 1$.

3. p-Adic strings

- p-Adic strings are strings which world-sheet is p-adic.
- Freund-Witten (adelic) product formula for ordinary and p-adic strings

$$A(a,b) = A_\infty(a,b) \prod_p A_p(a,b) = g_\infty^2 \prod_p g_p^2 = const.$$

- Ordinary and p-adic strings are at the equal footing
- Amplitude for real string $A_\infty(a,b)$, which is a special function, can be presented as product of inverse p-adic amplitudes, which are elementary functions.

3. p-Adic strings – Effective Lagrangian

$\geq$ 1988 : Freund, Witten, Frampton, Okada, ..., $\geq$ 2000 : Vladimirov, Volovich, Sen, Ghoshal, Zwiebach, Moeller, Schnabl, Barnaby, Kamran, Minahan, Yang, Biswas, Arefeva, Koshelev, Vernov, Joukovskaya, B.D., ...

- There is an effective field description of scalar open and closed p-adic strings. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher ones at the tree-level.
- The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \Big[-\frac{1}{2} \varphi \, p^{-\frac{\Box}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \Big]$$

where $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and metric with signature $(- + \ldots +)$.

3. p-Adic strings

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \Big[-\frac{1}{2} \varphi \, p^{-\frac{\Box}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \Big]$$

- Lagrangian with p-adic world-sheet B.D.

$$\mathcal{L}_p = \frac{m^D}{g^2} \frac{p^2}{p-1} \Big[\frac{1}{2} \varphi \int_{\mathbb{R}} \Big(\int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \chi_p(u) |u|_p^{\frac{k^2}{2m^2}} du \Big) \tilde{\varphi}(k) \, \chi_\infty(kx) \, d^4k + \frac{1}{p+1} \varphi^{p+1} \Big]$$

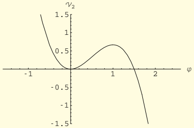
$$\int_{|x|_p > 1} \chi_p(u) |u|_p^s du = -p^s$$

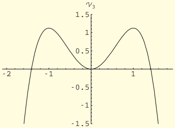
- The corresponding potential $\mathcal{V}(\varphi)$ for this Lagrangian is $\mathcal{V}_p(\varphi) = -\mathcal{L}_p(\Box = 0)$, which the explicit form is

$$\mathcal{V}_p(\varphi) = \frac{m^D}{g^2}\left[\frac{1}{2}\frac{p^2}{p-1}\varphi^2 - \frac{p^2}{p^2-1}\varphi^{p+1}\right].$$

It has local minimum $\mathcal{V}_p(0) = 0$. If $p \neq 2$ there are two local maxima at $\varphi = \pm 1$ and there is one local maximum $\varphi = +1$ when $p = 2$.

- The 2-adic string potential $\mathcal{V}_2(\varphi)$ (on the left) and 3-adic potential $\mathcal{V}_3(\varphi)$ (on the right) of the figure.





3. p-Adic strings – equation of motion

- The equation of motion is

$$p^{-\frac{\Box}{2m_p^2}}\,\varphi = \varphi^p$$

- It has trivial solutions $\varphi = 0$ and $\varphi = 1$, and $\varphi = -1$ for $p \neq 2$.
- There are also solutions in any direction x^i and time t

$$\varphi(x^i) = p^{\frac{1}{2(p-1)}}\ \exp\Big(-\frac{p-1}{2\,p\ln p}m^2(x^i)^2\Big)$$

$$\varphi(t) = p^{\frac{1}{2(p-1)}}\ \exp\Big(\frac{p-1}{2\,p\ln p}m^2t^2\Big)$$

3. p-Adic strings

- There is also D-dimensional solution

$$\varphi(x) = p^{\frac{D}{2(p-1)}} \exp\Big(-\frac{p-1}{2\,p \ln p} m^2 x^2 \Big), \quad x^2 = -t^2 + \sum_{i=1}^{D-1} x_i^2$$

- The above solutions can be obtained using identity

$$e^{A\,\partial_t^2}\, e^{B\,t^2} = \frac{1}{\sqrt{1-4AB}}\, e^{\frac{B\,t^2}{1-4AB}}, \quad 1-4AB > 0.$$

- p-Adic realization of tachyon condensation

3. p-Adic strings – Lagrangian for p-adic string sector

B.D.
Consider

$$L = \sum_{n=1}^{+\infty} C_n \, \mathcal{L}_n = m^D \sum_{n=1}^{+\infty} \frac{C_n}{g_n^2} \frac{n^2}{n-1} \Big[-\frac{1}{2} \, \phi \, n^{-\frac{\Box}{2m^2}} \, \phi + \frac{1}{n+1} \, \phi^{n+1} \Big]$$

where $\frac{C_n}{g_n^2} \frac{n^2}{n-1} = (-1)^{n-1}$. Take into account

$$\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s}) \, \zeta(s), \quad s = \sigma + i\tau \,, \quad \sigma > 0$$

which has analytic continuation to the entire complex s plane without singularities.

3. p-Adic strings – Lagrangian for p-adic string sector

- The corresponding Lagrangian is

$$L = m^D \Big[-\frac{1}{2}\phi\Big(1 - 2^{1-\frac{\Box}{2m^2}}\Big)\zeta\Big(\frac{\Box}{2m^2}\Big)\phi + \phi - \frac{1}{2}\log(1+\phi)^2\Big].$$

- The potential is

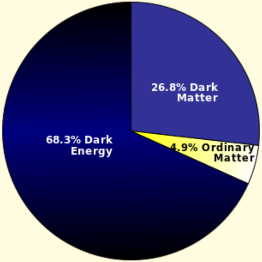
$$V(\phi) = -L(\Box = 0) = m^D\Big[\frac{1}{4}\phi^2 - \phi + \frac{1}{2}\log(1+\phi)^2\Big],$$

  which has one local maximum $V(0) = 0$ and one local minimum at $\phi = 1$. It is singular at $\phi = -1$.
- The equation of motion is

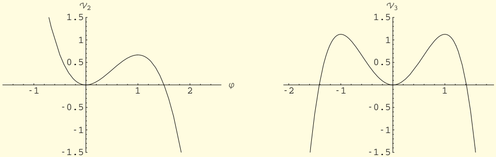
$$\Big(1 - 2^{1-\frac{\Box}{2m^2}}\Big)\zeta\Big(\frac{\Box}{2m^2}\Big)\phi = \frac{\phi}{1+\phi}, \quad \phi = 0, \ \ \phi = 1.$$

4. Possible gravitational and cosmological aspects

- If Einstein general theory of relativity is theory of gravity for the Universe as a whole, then there is only about 5% of ordinary matter, about 27% of dark matter and about 68% of dark energy.
- Conjecture: Dark matter and dark energy have *p*-adic origin.
- Modified gravity may be nonlocal.



- *p*-Adic inflation



5. Concluding remarks

- p-Adic numbers are not results of measurements, but can be useful in physical models.
- p-Adic strings are strings with world-sheet labeled by p-adic numbers.
- If there exist ordinary strings then there should exist also p-adic strings.
- p-Adic string theory is simpler than ordinary string theory, and seems to be useful for ordinary string theory.
- p-Adic strings may play significant role in description of dark side of the universe.
- Main applications in the near future in bioinformation systems.

5. Concluding remarks – some review references

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