On nonlocal modified gravity with cosmological solutions

Ivan Dimitrijević, Branko Dragovich, Jelena Grujić and Zoran Rakić

Motivation

Large cosmological observational findings:

- High orbital speeds of galaxies in clusters.(F.Zwicky, 1933)
- High orbital speeds of stars in spiral galaxies. (Vera Rubin, at the end of 1960es)
- Accelerated expansion of the Universe. (1998)

Problem solving approaches

There are two problem solving approaches:

- Dark matter and energy
- Modification of Einstein theory of gravity

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}, \; c = 1$$

where $T_{\mu\nu}$ is stress-energy tensor, $g_{\mu\nu}$ are the elements of the metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature of metric.

Dark matter and energy

- If Einstein theory of gravity can be applied to the whole Universe then the Universe contains about 4.9% of ordinary matter, 26.8% of dark matter and 68.3% of dark energy.
- It means that 95.1% of total matter, or energy, represents dark side of the Universe, which nature is unknown.
- Dark matter is responsible for orbital speeds in galaxies, and dark energy is responsible for accelerated expansion of the Universe.

Modification of Einstein theory of gravity

Motivation for modification of Einstein theory of gravity

- The validity of General Relativity on cosmological scale is not confirmed.
- Dark matter and dark energy are not yet detected in the laboratory experiments.
- Another cosmological problem is related to the Big Bang singularity. Namely, under rather general conditions, general relativity yields cosmological solutions with zero size of the universe at its beginning, what means an infinite matter density.
- Note that when physical theory contains singularity, it is not valid in the vicinity of singularity and must be appropriately modified.

Approaches to modification of Einstein theory of gravity

There are different approaches to modification of Einstein theory of gravity.

- Einstein General Theory of Relativity

From action $S = \int (\frac{R}{16\pi G} - L_m - 2\Lambda)\sqrt{-g}d^4x$ using variational methods we get field equations

$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$, $c = 1$

where $T_{\mu\nu}$ is stress-energy tensor, $g_{\mu\nu}$ are the elements of the metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature of metric. Currently there are mainly two approaches:

- f(R) Modified Gravity
- Nonlocal Gravity

Nonlocal Modified Gravity

Nonlocal gravity is a modification of Einstein general relativity in such way that Einstein-Hilbert action contains a function f(□, R). Our action is given by

$$S = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{16\pi G} + \frac{C}{2}R^p\mathcal{F}(\Box)R\Big)$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$ and C is a constant.

In the sequel we shall consider two nonlocal models separately for $p=+1$ and $p=-1$.

FLRW metric

We use Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\Big), \ k \in \{-1, 0, 1\}.$$

$$R = \frac{6\Big(a(t)\ddot{a}(t) + \dot{a}(t)^2 + k\Big)}{a(t)^2}$$

In case of FLRW metric the d'Alembert operator is

$$\Box R = -\ddot{R} - 3H\dot{R}, \qquad H = \frac{\dot{a}}{a}$$

Nonlocal Modified Gravity, $p = +1$

For $p = +1$ our action becomes

$$S = \int d^4x\sqrt{-g}\Big(\frac{R - 2\Lambda}{16\pi G} + \frac{C}{2}R\mathcal{F}(\Box)R\Big).$$

This model is attractive because it is ghost free and has some nonsingular bounce solutions, which can solve the Big Bang cosmological singularity problem.

Linear ansatz and nonsingular bounce cosmological solutions

Using ansatz $\Box R = rR + s, \quad r, s \in \mathbb{R}$ a few nonsingular bounce solutions for the scale factor are found:

- $a(t) = a_0 \cosh \sqrt{\frac{\Lambda}{3}} t$ for $k = 0$
- $a(t) = a_0 e^{\frac{1}{2}\sqrt{\frac{\Lambda}{3}} t^2}$ for $k = 0$

We have generalized the previous solutions. We found three types of nonsingular bouncing solutions for cosmological scale factor $a(t)$ in the form of a linear combination of $e^{\lambda t}$ and $e^{-\lambda t}$, i.e.

$$a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t}), \quad 0 < a_0, \lambda, \sigma, \tau \in \mathbb{R}$$

- All obtained solutions satisfy

$$\ddot{a}(t) = \lambda^2 a(t) > 0$$

- Solutions exist for all three values of spatial curvature constant $k = 0, \pm 1$.

Nonlocal Model with Term $R^{-1}\mathcal{F}(\Box)R$

For $p = -1$ and $C = 2$ our action becomes

$$S_1 = \int d^4x\sqrt{-g}\Big(\frac{R}{16\pi G} + R^{-1}\mathcal{F}(\Box)R\Big),$$

where $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$ and when $f_0 = -\frac{\Lambda}{8\pi G}$ it plays role of the cosmological constant.

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- The nonlocal term $R^{-1}\mathcal{F}(\Box)R$ is invariant under transformation $R \to CR$. It means that effect of nonlocality does not depend on the magnitude of scalar curvature R, but on its spacetime dependence, and in the FLRW case is sensitive only to dependence of R on time t.

Equations of Motion, $p = -1$

By variation of action S_1 with respect to metric $g^{\mu\nu}$ we obtain

$$
\begin{aligned}
&R_{\mu\nu} V - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) V - \frac{1}{2} g_{\mu\nu} R^{-1}\mathcal{F}(\Box) R \\
&+ \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Big(g_{\mu\nu} \left(\partial_\alpha \Box^l (R^{-1}) \partial^\alpha \Box^{n-1-l} R + \Box^l (R^{-1}) \Box^{n-l} R \right) \\
&- 2\partial_\mu \Box^l (R^{-1}) \partial_\nu \Box^{n-1-l} R \Big) = -\frac{G_{\mu\nu}}{16\pi G},
\end{aligned}
$$

where $V = \mathcal{F}(\Box) R^{-1} - R^{-2}\mathcal{F}(\Box) R$.

Equations of Motion, $p = -1$

The trace and 00-component of EOM are

$$RV + 3\Box V + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(\partial_\alpha \Box^l (R^{-1}) \partial^\alpha \Box^{n-1-l} R + 2\Box^l (R^{-1}) \Box^{n-l} R\right)$$
$$- 2R^{-1}\mathcal{F}(\Box)R = \frac{R}{16\pi G},$$

$$R_{00}V - (\nabla_0\nabla_0 - g_{00}\Box)V - \frac{1}{2} g_{00} R^{-1}\mathcal{F}(\Box)R$$
$$+ \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(g_{00}\left(\partial_\alpha \Box^l (R^{-1}) \partial^\alpha \Box^{n-1-l} R + \Box^l (R^{-1}) \Box^{n-l} R\right)\right.$$
$$\left. - 2\partial_0 \Box^l (R^{-1}) \partial_0 \Box^{n-1-l} R\right) = -\frac{G_{00}}{16\pi G}.$$

Cosmological solutions with constant scalar curvature

Let $R = R_0 = const$ and we obtain

$$6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) = R_0.$$

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Depending on the sign of R_0 we have the following solutions for $b(t)$

$$R_0 > 0 \quad b(t) = \frac{6k}{R_0} + \sigma e^{\sqrt{\frac{R_0}{3}}t} + \tau e^{-\sqrt{\frac{R_0}{3}}t},$$

$$R_0 < 0 \quad b(t) = \frac{6k}{R_0} + \sigma \cos\sqrt{\frac{-R_0}{3}}t + \tau \sin\sqrt{\frac{-R_0}{3}}t.$$

Cosmological solutions with constant scalar curvature

When we set $R = R_0 = const$ into trace and 00-equation we obtain the following system

$$-2f_0 = \frac{R_0}{16\pi G}, \qquad \frac{1}{2}f_0 = -\frac{G_{00}}{16\pi G}$$

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The last system has a solution iff

$$R_0 + 4R_{00} = 0.$$

Note that R_{00} can be written in terms of function $b(t)$ as

$$R_{00} = -\frac{3\ddot{a}}{a} = \frac{3((\dot{b})^2 - 2b\ddot{b})}{4b^2}.$$

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Now, from $R_0 + 4R_{00} = 0$ we obtain the following conditions on the parameters σ and τ:

$$\begin{aligned} R_0 > 0 \qquad & 9k^2 = R_0^2 \sigma\tau, \\ R_0 < 0 \qquad & 36k^2 = R_0^2(\sigma^2 + \tau^2). \end{aligned}$$

Case 1: $R_0 < 0$

If $k = -1$ we can define φ by $\sigma = \frac{-6}{R_0}\cos\varphi$ and $\tau = \frac{-6}{R_0}\sin\varphi$ and rewrite $a(t)$ and $b(t)$ as

$$b(t) = \frac{-12}{R_0}\cos^2\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t - \varphi\right),$$

$$a(t) = \sqrt{\frac{-12}{R_0}}\left|\cos\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t - \varphi\right)\right|.$$

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In the last case $k = +1$ we can transform $b(t)$ to

$$b(t) = \tfrac{12}{R_0}\sin^2\tfrac{1}{2}\Big(\sqrt{-\tfrac{R_0}{3}}\,t - \varphi\Big),$$

which is non positive and hence yields no solutions.

Case 2: $R_0 > 0$

Set $k = 0$ then we obtain a solution with constant Hubble parameter. Alternatively, if we set $k = +1$ we can find φ such that $\sigma + \tau = \frac{6}{R_0} \cosh \varphi$ and $\sigma - \tau = \frac{6}{R_0} \sinh \varphi$. Moreover, we obtain

$$b(t) = \frac{12}{R_0} \cosh^2 \frac{1}{2}\Big(\sqrt{\frac{R_0}{3}}\, t + \varphi\Big),$$

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At the end if we set $k = -1$ we can transform $b(t)$ to

$$b(t) = \frac{12}{R_0}\sinh^2\frac{1}{2}(\sqrt{\frac{R_0}{3}}t + \varphi),$$

$$a(t) = \sqrt{\frac{12}{R_0}}|\sinh\frac{1}{2}(\sqrt{\frac{R_0}{3}}t + \varphi)|.$$

Case 2.1: $R = 12\lambda^2$

We consider the scale factor of the form

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We have

$$H(t) = \frac{\lambda(e^{2\lambda t}\sigma_1 - \tau_1)}{e^{2\lambda t}\sigma_1 + \tau_1},$$
$$R(t) = \frac{6(e^{2\lambda t}k + 2\lambda^2(e^{4\lambda t}\sigma_1^2 + \tau_1^2)a_0^2)}{(e^{2\lambda t}\sigma_1 + \tau_1)^2 a_0^2}.$$

In order to have $R = const$ we have to satisfy condition
$k = 4\lambda^2 a_0^2 \sigma_1 \tau_1$.
From the last condition we obtain $R = 12\lambda^2$.

Case 2.1: $R = 12\lambda^2$

Substituting this into equations trace and 00-component we obtain

$$f_0 = -\frac{3\lambda^2}{8\pi G}, \quad f_i \in \mathbb{R}, \quad i \geq 1.$$

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In particular, if we set $\sigma_1 = \tau_1 = \frac{1}{2}$ the scale factor becomes

$$a(t) = a_0 \cosh(\lambda t).$$

In this case from condition $k = 4\lambda^2 a_0^2 \sigma_1 \tau_1$ we see that the only nontrivial case is when k is equal to 1. From this we obtain $a_0 = \frac{1}{\lambda}$.

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In this case from condition $k = 4\lambda^2 a_0^2 \sigma_1 \tau_1$ we see that the only nontrivial case is when k is equal to 1. From this we obtain $a_0 = \frac{1}{\lambda}$. If we take $\sigma_1 = 0$ or $\tau_1 = 0$ the scale factor becomes

$$a(t) = a_0 e^{\lambda t}.$$

From condition $k = 4\lambda^2 a_0^2 \sigma_1 \tau_1$ we see that in this case k must be equal to 0.

Case 3: $R_0 = 0$

- The case $R_0 = 0$ can be considered as limit of $R_0 \to 0$ in both cases $R_0 < 0$ and $R_0 > 0$.

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- When $R_0 < 0$ there is condition $36k^2 = R_0^2(\sigma^2 + \tau^2)$. From this condition, $R_0 \to 0$ implies $k = 0$ and arbitrary values of constants σ and τ. The same conclusion obtains when $R_0 > 0$ with condition $9k^2 = R_0^2\sigma\tau$. In both these cases there is Minkowski solution with $b(t) = constant > 0$ and consequently $a(t) = constant > 0$.

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- Note that the Minkowski space solution can be also obtained from the case $R = 12\lambda^2$. Namely, the solution $a(t) = a_0 e^{\lambda t}$ satisfies $H = \lambda$. Taking the limit $\lambda \to 0$ in $a(t) = a_0 e^{\lambda t}$ one obtains Minkowski space as a solution for

$$f_0 = 0, \quad f_i \in \mathbb{R}, \quad i \geq 1.$$

Concluding remarks, $p = -1$

- We presented cosmological solutions for constant scalar curvature of model given by

$$S_1 = \int d^4x\sqrt{-g}\Big(\frac{R}{16\pi G} + R^{-1}\mathcal{F}(\Box)R\Big).$$

- When $R = R_0 < 0$ there is nontrivial solution $a(t) = \sqrt{\frac{-12}{R_0}}\left|\cos\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t - \varphi\right)\right|$ for $k = -1$.
- In the case $R = R_0 > 0$ there are solutions for all three values of curvature constant $k = 0, \pm 1$.
- The case $R = R_0 = 0$ was considered as limit of $R_0 \to 0$ in both cases $R_0 < 0$ and $R_0 > 0$, and Minkowski space solution was obtained.
- All obtained solutions are defined for all values of cosmic time t.
- Solutions for $R_0 > 0$ with $k = 0, +1$ are nonsingular bounce cosmological solutions.
- Solution $a(t) = \sqrt{\frac{-12}{R_0}}\left|\cos\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t - \varphi\right)\right|$, which is for $R_0 < 0$ and $k = -1$, is a singular cyclic solution.

For details see

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- Dimitrijevic, I., Dragovich, B., Grujic J., Rakic, Z.: New cosmological solutions in nonlocal modified gravity. Rom. Journ. Phys. 58 (5-6), 550559 (2013) [arXiv:1302.2794 [gr-qc]]
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