Higher-Dimensional Theories with continuous or fuzzy coset spaces as extra dimensions

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ABSTRACT

We review the Coset Space Dimensional Reduction (CSDR) scheme and the best model constructed so far. Then we present some details of the CSDR programme, in an alternate version, in which the extra dimensions are considered to be fuzzy. Specifically, we present a four-dimensional $\mathcal{N} = 4$ SYM theory, orbifolded by $\mathbb{Z}_3$, which mimics the behaviour of a dimensionally reduced $\mathcal{N} = 1$, ten-dimensional gauge theory over a set of fuzzy spheres at intermediate high scales. This leads to the trinification GUT $SU(3)^3$ at slightly lower, which in turn can be spontaneously broken to the MSSM in low scales.

1. Introduction

Scientists have set in high priority the aspect of unification of the fundamental forces. Appealing approaches are the ones that support the existence of extra dimensions. A very consistent framework in the unification of all forces supporting such a scenario is Superstring theories [1], with most promising the Heterotic string [2] due to the connections to the low-energy physics. Another remarkable framework for the unification attempt was employed, a few years before the discovery of the Heterotic string, that is the dimensional reduction of higher-dimensional gauge theories. This field was pioneered by Forgacs and Manton with studies on Coset Space Dimensional Reduction (CSDR) [3, 4, 5] and the Scherk-Schwarz group manifold reduction [8]. In these two approaches, a starting gauge theory governs the regime of higher dimensions, where gauge-Higgs unification is achieved, leading to a four-dimensional theory in which the gauge and Higgs fields are the surviving components of the initial fields in high dimensions. In the CSDR scheme, fermions are included in the initial gauge

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theory, resulting to Yukawa couplings in four dimensions. The initial theory is required to be $\mathcal{N} = 1$ supersymmetric, i.e. gauge and fermion fields belong to the same vector supermultiplet, relating gauge and fermion fields that have been introduced. Resulting with chiral theories in four dimensions [9, 10] is regarded as a notable achievement.

In order to preserve an $\mathcal{N} = 1$ supersymmetry after the dimensional reduction, Calabi-Yau (CY) spaces are considered as suitable compact internal manifolds [11]. However, the moduli stabilization problem that arose, led to a wider class of internal spaces, called manifolds with SU(3)-structure. Specifically, here we consider an interesting class of SU(3)-structure manifolds, called nearly-Kähler manifolds [12, 14, 15], also see refs from [13].

The homogeneous nearly-Kähler manifolds in six dimensions are the three non-symmetric coset spaces $G_2/SU(3)$, $Sp(4)/(SU(2) \times U(1))_{non-max}$ and $SU(3)/U(1) \times U(1)$ and the group manifold $SU(2) \times SU(2)$ [15] (see also [14, 12]). It is worth noting that in four-dimensional theories resulting from dimensional reduction of a ten-dimensional, $\mathcal{N} = 1$ supersymmetric gauge theory over non-symmetric coset spaces, supersymmetry breaking terms are automatically included [16], [17], contrary to CY spaces.

Another promising framework for describing physics at Planck scale is Non-commutative geometry [18] - [38]. Non-commutative geometry was considered as an appropriate framework for regularizing quantum field theories, or even better, building finite ones. However, constructing quantum field theories on Non-commutative spaces is much more difficult than expected and , furthermore, problematic ultraviolet features have emerged [21] (see also [22] and [23]). Nevertheless, this framework is appropriate to accommodate particle models with Non-commutative gauge theory [24] (see also [25, 26, 27]).

Remarkably, the two frameworks came closer by realizing that in M-theory and "open string theory", in the presence of a non-vanishing background antisymmetric field, the effective physics on D-branes can be described by an Non-commutative gauge theory [28, 29]. Thus, Non-commutative field theories emerge as effective description of string dynamics. Moreover, major contribution in Non-commutative geometry was made by Seiberg and Witten [29]. Their study triggered notable developments [32, 31] and , based on them, Non-commutative versions of SM were built [33]. Unfortunately, those models fail to troubleshoot the main problem of the SM, that is the presence of numerous free parameters, due to the ad hoc consideration of Higgs and Yukawa sectors. Finally, an interesting programme has been suggested and investigated [34, 35, 36, 37, 38] considering the extra dimensions as Non-commutative. This programme overcomes the ultraviolet/infrared problems of theories defined in Non-commutative spaces in an obvious way offering the new possibility to start with an abelian gauge theory defined on the higher-dimensional space and result with a non-abelian one in four dimensions, after dimensional reduction. Additionally, another spectacular feature of this programme is that theories constructed on Non-commutative (fuzzy) manifolds as approximations of the continuous ones, are renormalizable contrary to all known higher-dimensional theories. The latter property was examined from the four-dimensional point of view, too,

using spontaneous symmetry breakings, which mimic the results of the dimensional reduction of a higher-dimensional gauge theory with fuzzy extra dimensions. Finally, in this framework, chiral realistic theories have been constructed, too.

2. The Coset Space Dimensional Reduction

The CSDR procedure demands that the field dependence on the extra coordinates is such that the Lagrangian is independent of them. An elegant way to fulfill the above requirement is to allow for a non-trivial dependence on them, in the sense that a symmetry transformation by an element of the isometry group S of the space formed by the extra dimensions B corresponds to a gauge transformation. Along this framework, a gauge invariant Lagrangian will be independent of the extra coordinates. The above mechanism is the basis of the CSDR scheme [3, 4, 5], which assumes that B is a compact coset space, S/R.

In the CSDR scheme one considers a Yang-Mills-Dirac Lagrangian, with gauge group G, defined on a D-dimensional spacetime M^D, with metric g^{MN}, which is compactified to $M^4 \times S/R$, with S/R a coset space. We assume the following form for the metric

$$g^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & -g^{ab} \end{pmatrix} , \tag{1}$$

where $\eta^{\mu\nu} = diag(1, -1, -1, -1)$ and g^{ab} is the coset space metric. The requirement that transformations of the fields under the action of the symmetry group of S/R are compensated by gauge transformations, imposes certain constraints on the fields of our theory. The analysis of these constraints provides us with the four-dimensional unconstrained fields, as well as with the gauge invariance that remains in the theory after dimensional reduction. Therefore, a potential unification of all low energy interactions, gauge, Yukawa and Higgs is achieved.

It is worth noting that the dimensional reduction of higher-dimensional theories results in effective field theories that might contain also towers of massive higher harmonic (Kaluza-Klein) excitations. The behaviour of the running couplings is altered from logarithmic to power [42] by the quantum level contributions of these excitations, resulting in a remarkable change of the traditional unification picture [43]. Using the continuous Wilson renormalization group technique [44], which can be formulated in any number of space-time dimensions, higher-dimensional theories have also been studied at the quantum level, with results in agreement with the treatment involving massive Kaluza-Klein excitations.

2.1. Reduction of a D-dimensional Yang-Mills-Dirac Lagrangian.

Considering a Lie group S and its subgroup R we define a d-dimensional coset S/R on which the extra dimensions of $M^4 \times S/R$ are compactified (M^4 is our space-time). S acts as a symmetry group of the extra coordinates.

According to the CSDR scheme, an S-transformation of the extra d coordinates is a gauge transformation of the fields that are defined on $M^4 \times S/R$, thus a gauge invariant Lagrangian written on this space is independent of the extra coordinates. Fields defined in this way are called symmetric. The d-dimensional gauge field $A_M(x,y)$ is split into its components $A_\mu(x,y)$ and $A_\alpha(x,y)$, corresponding to M^4 and S/R respectively.

Let us now consider a Yang-Mills-Dirac theory with gauge group G defined on a manifold M^D which, as stated, will be compactified to $M^4 \times S/R$, $D = 4 + d$, $d = dimS - dimR$. The action is

$$A = \int d^4x d^d y \sqrt{-g} \left[-\frac{1}{4} Tr(F_{MN} F_{K\Lambda}) g^{MK} g^{N\Lambda} + \frac{i}{2} \bar{\psi} \Gamma^M D_M \psi \right] , \qquad (2)$$

where $D_M = \partial_M - \theta_M - A_M$, with $\theta_M = \frac{1}{2}\theta_{MN\Lambda}\Sigma^{N\Lambda}$, the spin-connection of M^D, and $F_{MN} = \partial_M A_N - \partial_N A_M - [A_M, A_N]$, where M, N run over the D-dimensional space and A_M and ψ are D-dimensional symmetric fields. The fermion fields can be accommodated in any representation F of G, unless a further symmetry, such as supersymmetry, is required. If we denote by ξ_A^α, $(A = 1, ..., dimS$ and $\alpha = dimR + 1, ..., dimS$ the curved index) the Killing vectors which generate the symmetries of S/R and by W_A the compensating gauge transformation associated with ξ_A, the following constraint equations for scalar ϕ, vector A_α and spinor ψ fields on S/R, are expressing the requirement that transformations of the fields under the action of S/R are compensated by gauge transformations

$$\delta_A \phi = \xi_A^\alpha \partial_\alpha \phi = D(W_A)\phi, \qquad (3)$$

$$\delta_A A_\alpha = \xi_A^\beta \partial_\beta A_\alpha + \partial_\alpha \xi_A^\beta A_\beta = \partial_\alpha W_A - [W_A, A_\alpha], \qquad (4)$$

$$\delta_A \psi = \xi_A^\alpha \partial_\alpha \psi - \frac{1}{2} G_{Abc} \Sigma^{bc} \psi = D(W_A)\psi , \qquad (5)$$

where W_A depend only on internal coordinates y and $D(W_A)$ represents a gauge transformation in the appropriate representation of the fields.

Regarding the constraints (3)-(5), they provide us [3, 6, 4] with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction. The components $A_\mu(x,y)$ of the initial gauge field $A_M(x,y)$ become, after dimensional reduction, the four dimensional gauge fields and they are independent of y. Additionally, they have to commute with the elements of the R_G, subgroup of G, meaning that the four-dimensional gauge group H is the centralizer of R in G, $H = C_G(R_G)$. We denote by $\phi_\alpha(x,y)$ the $A_\alpha(x,y)$ components of $A_M(x,y)$. They become scalars in four dimensions and they transform under R as a vector v, i.e.

$$S \supset R \qquad (6)$$

$$adjS = adjR + v. \qquad (7)$$

Moreover, the $\phi_\alpha(x,y)$ fields act as an interwining operators connecting induced representations of R acting on G and S/R. According to Schur's lemma, the previous expression implies that the transformation properties of the fields $\phi_\alpha(x,y)$ under H can be found, if we decompose the adjoint representation of G according to the embedding:

$$G \supset R_G \times H \tag{8}$$

$$adjG = (adjR, 1) + (1, adjH) + \sum (r_i, h_i). \tag{9}$$

Then, if $\upsilon = \sum s_i$, where each s_i is an irreducible representation of R, there survives a Higgs multiplet transforming under the representation h_i of H and all other scalar fields vanish.

Regarding the fermion fields [4, 45, 9, 10] we proceed along similar lines as in the case of scalars. It turns out that the spinor fields act as interwining operators connecting induced representations of R in $SO(d)$ and G. In order to obtain the H representation content of the four-dimensional fermions, we have to decompose the representation F of the initial gauge group, in which the fermions are assigned in higher dimensions, under $R_G \times H$, i.e.

$$F = \sum (t_i, h_i), \tag{10}$$

and the spinor of $SO(d)$ under R

$$\sigma_d = \sum \sigma_j. \tag{11}$$

Then for each pair (r_i, σ_i), where r_i and σ_i are identical irreducible representations of R there is an h_i multiplet of spinor fields in the four-dimensional theory. Regarding the possibility of obtaining chiral fermions in the effective theory, we notice that if we start with Dirac fermions in higher dimensions it is impossible to obtain chiral fermions in four dimensions. Further requirements must be considered in order to achieve chiral fermions in the resulting theory. Imposing the Weyl condition in D dimensions, we obtain two sets of Weyl fermions with the same quantum numbers under H. Although this is already a chiral theory, we can go further and try to impose Majorana condition in order to eliminate the doubling of the fermionic spectrum. Majorana and Weyl conditions are compatible in $D = 4n + 2$ which is the case of our interest.

The allowed embeddings of R into G are restricted by the condition that an anomaly free theory in higher dimensions must fulfill, in order to obtain anomaly free theories in four dimensions after the dimensional reduction [46]. According to that condition, the allowed embeddings are related with the embedding of R into $SO(6)$, the tangent space of the six-dimensional cosets we consider [4, 40, 7]. According to ref. [7] the anomaly cancelation condition is automatically satisfied for the choice of embedding $E_8 \supset SO(6) \supset R$, which we adopt here.

2.2. Dimensional Reduction of E_8 over $SU(3)/U(1) \times U(1)$

In this subsection we summarize a few results concerning the dimensional reduction of the $\mathcal{N} = 1, E_8$ SYM over $SU(3)/U(1) \times U(1)$ [39]. The four-dimensional gauge group will be provided by the decomposition of E_8 under $R = U(1) \times U(1)$ suggested by

$$E_8 \supset E_6 \times SU(3) \supset E_6 \times U(1)_A \times U(1)_B \,. \tag{12}$$

According to the rules of the previous section, the surviving gauge group in four dimensions is

$$H = C_{E_8}(U(1)_A \times U(1)_B) = E_6 \times U(1)_A \times U(1)_B \,. \tag{13}$$

The surviving scalars and fermions in four dimensions are provided by the explicit decomposition of the adjoint representation of E_8, 248 under $U(1)_A \times U(1)_B$. Applying the CSDR rules we find that the resulting four-dimensional theory is an $\mathcal{N} = 1$, E_6 GUT with $U(1)_A, U(1)_B$ as global symmetries. The potential is determined by the decomposition of the specific $S = SU(3)$ under $R = U(1) \times U(1)$ studied in [17]. The D-terms can be constructed and the F-terms are obtained by the superpotential. The rest of the terms in the potential could be interpreted as soft scalar masses and trilinear soft terms. Finally, the gaugino obtains a mass and receives contribution from the torsion contrary to the rest soft supersymmetry breaking terms.

2.3. $SU(3)^3$ due to Wilson flux

According to the previous section, the $E_6 \times U(1) \times U(1)$ group is the surviving gauge group of the initial's E_8 group dimensional reduction. The surviving scalars in the four-dimensional theory, being in the fundamental representation of the gauge group are not able to provide the appropriate symmetry breaking towards the standard model. In order to reduce further the gauge symmetry, one has to apply the Wilson flux breaking mechanism [49, 48, 47]. Application of this mechanism imposes further constraints in the scheme.

In the case of our interest, instead of considering the simply connected manifold B_0, where B_0 is the coset S/R, we consider the multiply connected manifold $B = B_0/F^{S/R}$ with $F^{S/R}$ a freely acting discrete symmetry of B_0. The manifold B is multiply connected due to the presence of the symmetry $F^{S/R}$. For each element $g \in F^{S/R}$, we pick up an element U_g in H, which can be represented as the Wilson loop. If the manifold is simply connected, then the vanishing of the field strength ensures that we can set the gauge field to zero by a gauge transformation. In the case of a multiply connected manifold, although the vacuum field strength vanishes everywhere, U_g cannot be set to one and the gauge field cannot be set to zero. Therefore, a homomorphism of $F^{S/R}$ into H is induced with image T^H, which is the subgroup of H generated by the element U_g.

Concerning the gauge symmetry that is preserved by the vacuum, we consider the following. The vacuum has $A_\mu^a = 0$ and we represent a gauge

transformation by a space-dependent matrix $V(x)$ of H. In order to keep $A_\mu^a = 0$ and leave the vacuum invariant, $V(x)$ must be a constant. Moreover, the matrix $V(x)$ is consistent with the action of the elements U_g only if $[V, U_g] = 0$ for all $g \in F^{S/R}$. Therefore, the unbroken subgroup of H is the centralizer of T^H in H. Respectively, the surviving matter fields are those that are invariant under the diagonal sum $F^{S/R} \oplus T^H$. The discrete symmetries $F^{S/R}$, which act freely on coset spaces $B_0 = S/R$ are the center of S, $Z(S)$ and $W = W_S/W_R$, where W_S and W_R are the Weyl groups of S and R, respectively. In the case of our interest, where $B_0 = SU(3)/U(1) \times U(1)$, we have $F^{S/R} = \mathbb{Z}_3 \subseteq W$. After the $\mathbb{Z}_3$ projection, the gauge group E_6 breaks to $SU(3)_C \times SU(3)_L \times SU(3)_R$, (the first of the $SU(3)$ factors is the Standard Model colour gauge group). Moreover, one can obtain three fermion generations by introducing non-trivial monopole charges in the $U(1)$'s in R.

In ref [14] it was shown that the scalar potential leads to the proper hierarchy of spontaneous breaking. Using the appropriate vev's, a first spontaneous symmetry breaking leads to the MSSM [50], while the electroweak breaking proceeds by a second one [51]. It is worth noting that before the EW symmetry breaking, supersymmetry is broken by both D-terms and F-terms, in addition to its breaking by the soft terms.

We plan to examine in detail the phenomenological consequences of the resulting model, taking also into account the massive Kaluza-Klein modes.

3. Fuzzy spaces and fuzzy dimensional reduction

In order to continue our analysis, it is fundamental to introduce the concept of the fuzzy sphere [41]. The appropriate way to do so, is to initially consider the ordinary sphere S^2, on which the algebra of functions is commutative, and then define the fuzzy sphere as its extension.

It is known that the algebra of functions on S^2 is generated by the spherical harmonics, Y_{lm}, i.e. any arbitrary function on S^2 can be expanded in terms of Y_{lm}, since they form a complete and orthogonal set of functions. In the fuzzy case (the most typical case of Non-commutative geometry), contrary to the non-fuzzy sphere, the integer number l does have an upper limit. So, the algebra of functions on the fuzzy sphere is truncated to finite dimensional - naturally considered as a matrix algebra. Therefore, it proves that it is consistent to define the fuzzy sphere[1] as a matrix approximation of the non-fuzzy sphere and that the truncation of the algebra of the functions[2] is responsible for the deprivation of commutativity.

Given that we aim at studying gauge theory on fuzzy sphere, the next -obvious- step is to examine the behaviour of the gauge fields on the fuzzy sphere. So, we consider a field $\phi(X_a)$ on the fuzzy sphere, with X_a being

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[1]The geometry of other (higher-dimensional) fuzzy spaces (e.g. fuzzy CP^M) are examined in [52, 53]

[2]Besides of functions on the fuzzy sphere, spinors can be examined as well [34]

the covariant coordinates [54] and then we take an infinitesimal transformation of this field

$$\delta\phi(X) = \lambda(X)\phi(X)\,, \tag{14}$$

where $\lambda(X)$ is the parameter of the gauge transformation. Under the above gauge transformation it holds that $\delta X_a = 0$, ensuring the invariance of the covariant coordinates. Therefore, in the Non-commutative case, when left multiplying by a coordinate, we obtain

$$\delta(X_a\phi) = X_a\lambda(X)\phi\,, \tag{15}$$

and in general, it holds that $X_a\lambda(X)\phi \neq \lambda(X)X_a\phi$. So, according to the non-fuzzy gauge theory, one needs to introduce the covariant coordinates phi_a, in order to obtain $\delta(\phi_a\phi) = \lambda\phi_a\phi$, with $\delta(\phi_a) = [\lambda, \phi_a]$. Also, it is set that $\phi_a \equiv X_a + A_a$, where A_a is the gauge potential, concluding in the equivalence that ϕ_a is the analogue of the covariant derivative of the original gauge theories. From the above equations, the transformation of A_a is

$$\delta A_a = -[X_a, \lambda] + [\lambda, A_a]\,, \tag{16}$$

encouraging the identification of A_a with a gauge field.

4. Fuzzy CSDR

Attempts to reproduce the dominant gauge theory that describes physics in low energies are based on the above structure. More specifically, we consider a Non-commutative gauge theory on the $M^4 \times (S/R)_F$[3] space, then we perform dimensional reduction and in the end we result with a four dimensional theory. Unfortunately, realistic results did not arise using this method, therefore, in order to obtain a more appropriate gauge theory in four dimensions, a non trivial dimensional reduction had to be applied, namely the fuzzy extension of the CSDR scheme.

The factor that differentiates the fuzzy CSDR from the original one, is the consideration of the extra dimensions as fuzzy coset spaces [34] (see also [55]), meaning that the group S acts now on the fuzzy coset $(S/R)_F$, with the fields remaining invariant under an infinitesimal transformation of S - up to an infinitesimal gauge transformation. Specifically, the fuzzy coset we make use is the fuzzy sphere, $(SU(2)/U(1))_F$, therefore scalar and gauge fields should be left invariant under an infinitesimal transformation of $SU(2)$ on the fuzzy sphere, up to an infinitesimal gauge transformation

$$\mathcal{L}_b\phi = \delta_{W_b} = W_b\phi \tag{17}$$

$$\mathcal{L}_b A = \delta_{W_b} A = -DW_b\,, \tag{18}$$

where A is the gauge potential and W_b is an antihermitian gauge parameter which depends on X_a. Therefore, W_b can be written as

$$W_b = W_b^I \mathcal{T}^I\,, \quad I = 1, 2, \ldots, P^2\,, \tag{19}$$

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[3]$(S/R)_F$ is a fuzzy coset e.g. the fuzzy sphere.

where $\mathcal{T}^I$ are the hermitian generators of the gauge group of the theory $U(P)$ and $(W_b^I)^\dagger = -W_b^I$. The CSDR constraints are converted in the form

$$[\omega_b, A_\mu] = 0 \qquad (20)$$

$$C_{bde}\phi^e = [\omega_b, \phi_d], \qquad (21)$$

where $\phi_a \equiv X_a + A_a$ -as mentioned above- and $\omega_a \equiv X_a - W_a$. Since Lie derivatives respect the $\mathfrak{su}(2)$ commutation relations, one is led to the consistency condition

$$[\omega_a, \omega_b] = C_{ab}^c \omega_c, \qquad (22)$$

where ω_a transforms as $\omega_a \to \omega_a' = g\omega_a g^{-1}$. As for the spinors, a quite similar procedure that is followed [34].

As an application of the fuzzy CSDR scheme, we present the example, where the gauge group is $U(1)$ and the fuzzy coset is the fuzzy sphere. The ω_a are $N \times N$ antihermitian matrices, therefore they can be considered as elements of $U(N)$. Though, the consistency relation, (22), must hold, that is the ω_a obeys the commutation relation of the $SU(2)$ algebra. Thus, the $SU(2)$ algebra has to be embedded into the $U(1)$ algebra. So, let $T^h, h = 1 \ldots, N^2$ be the generators of the $U(N)$ in the fundamental representation, and make use of the convention $h(a, u), a = 1, 2, 3, u = 4, 5, \ldots, N^2$, with the generators T^a satisfying the $SU(2)$ algebra $[T^a, T^b] = C_c^{ab} T^c$. Obviously the embedding is achieved with the identification $\omega_a = T_a$.

Let us now examine and give interpretations of the two constraints (20), (21). The first one suggests that the gauge group of the four-dimensional theory is the centralizer of the image of $SU(2)$ in $U(N)$, that is

$$K = C_{U(N)}(SU(2)) = SU(N-2) \times U(1) \times U(1). \qquad (23)$$

Therefore, there is an arbitrariness on the dependance of $A_\mu(x, X)$ on x, but as for X, they depend on them meaning that the latter are valued in K instead of $U(N)$. Rephrasing, the 4-dimensional gauge potential that one is led is valued in K. The second constraint is satisfied after choosing $\phi_a = r\phi(x)\omega_a$. This means that the remaining unconstrained degrees of freedom are related to the scalar field $\phi(x)$, which belongs to the trivial representation of the 4-dimensional gauge group K.

Summing up the above procedure, one starts with a gauge theory which is described by a $U(1)$ on $M^4 \times S_N^2$. The consistency condition is satisfied by embedding the $SU(2)$ into $U(N)^4$. Then, imposing the two CSDR constraints, the four-dimensional group is obtained and the scalar fields that do survive the reduction procedure arise.

Let us now proceed with listing the results of the above procedure, for the fermionic case. The extended analysis [34] proves that the appropriate embedding is $S \subset SO(\dim S)$, which is achieved by $T_a = \frac{1}{2}C_{abc}\Gamma^{bc}$, which respects the $SU(2)$ commutation relation. Therefore, ψ is an interwining

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[4]Instead of embedding the $SU(2)$ into the fundamental representation of $U(N)$, one could have used other representations, too [41]

operator between the representations of S and $SO(\dim S)$. According to the commutative case [4], the surviving fermions in four-dimensional theory arise by decomposing the adjoint representation of $U(NP)$ under the product $S_{U(NP)} \times K$, that is

$$U(NP) \supset S_{U(NP)} \times K \,, \quad \mathrm{adj}\, U(NP) = \sum_i (s_i, k_i) \,. \qquad (24)$$

Moreover, the decomposition of the spinorial representation σ of $SO(\dim S)$ under S is

$$SO(\dim S) \supset S \,, \quad \sigma = \sum_e \sigma_e \,. \qquad (25)$$

Therefore, in case that the two irreducible representations s_i, σ_e are identical, the fermions that survive (4-dimensional spinors) and are present in the four-dimensional theory, belong to the k_i representation of K.

Ending this section, it is important to compare the ordinary higher-dimensional theory $M^4 \times (S/R)$, to the fuzzy one, $M^4 \times (S/R)_F$. Both theories have the same isometries - fuzziness does not affect them -, i.e. $SO(1,3) \times SO(3)$. In addition, the dimensionality of the gauge couplings defined on the two spaces is the same. On the other side, they present a very striking difference: Non-commutative higher-dimensional theory is the only one that is renormalizable[5]. Moreover, a $U(1)$ gauge group defined on the $M^4 \times (S/R)_F$ space, is appropriate in order to end up with a non-abelian four-dimensional theory [6].

5. Orbifolds and fuzzy extra dimensions

The recovering of chiral four-dimensional theories starting from higher-dimensional theories with fuzzy extra dimensions was the motivation of the introduction of the orbifold structure, similar to the one in [56]. The orbifold procedure offers an alternative way to obtain $\mathcal{N} = 1$ four-dimensional models after reducing a higher-dimensional theory on appropriate manifolds, e.g. Calabi-Yau [57] or $SU(3)$ structured ones [12, 58]. Duality between four-dimensional $\mathcal{N} = 4$, $U(N)$ SYM theory and Type IIB string theory on $AdS_5 \times S^5$ [59], motivated the authors of [56] to proceed to the application of orbifold techniques - similar to [60, 61] - in order to break some of the four supersymmetries. Moreover, the initial gauge group, $SU(3N)$, that is realized on $3N$ $D3$ branes[7], breaks to $SU(N)^3$ with fermions being accommodated into its chiral representations.

The concept of deconstructing dimensions [62], motivated the idea to reverse the above procedure [35, 36, 37] for further justification of the

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[5]Meaning that the divergencies are eliminated by a finite number of counter-terms.

[6]Technically, this is possible because $N \times N$ matrices could be decomposed on the $U(N)$ generators.

[7]This is point where the two different frameworks (superstring theories and Non-commutative geometry) that aim at unification meet, i.e. Non-commutative gauge theory can describe effective physics on D-branes.

renormalizability of the theory and construction of chiral models in theories arising from the framework of fuzzy extra dimensions. Reversing the procedure gives hope that consideration of the initial abelian gauge theory as a higher-dimensional one is not necessary, instead the non-abelian gauge theory could emerge from fluctuations of the coordinates [63]. Realizing the last consideration, one has to start with a four-dimensional gauge theory, including an appropriate scalar spectrum and a suitable potential producing vacua that could be interpreted as dynamically generated fuzzy extra dimensions including, at the same time, a finite Kaluza-Klein tower of massive modes. Also, although in such models the inclusion of chiral fermions is preferred, the best one achieved so far includes mirror fermions[8] [36, 37].

In this review, the above sketch is realized performing a dimensional reduction on an orbifold [65, 64]. More specifically, we examine the spectrum of the surviving fields and the superpotential of the projected theory, after the application of $\mathbb{Z}_3$ orbifold projection of the $\mathcal{N} = 4$ SYM theory [66]. In our case, this theory is an $SU(3N)$ and the particle content is one $SU(3N)$ gauge supermultiplet and three adjoint chiral supermultiplets. Their component fields are the gauge bosons, six adjoint real scalars and four adjoint Weyl fermions. The scalars and Weyl fermions transform under representations of the $SU(4)_R$ symmetry of the theory - 6, 4 respectively. That is the reason why - in order to introduce orbifolds - the discrete group $\mathbb{Z}_3$ must be included as a subgroup of $SU(4)_R$. Although there are more than one options, the appropriate one is to embed the discrete group into the $SU(3)$ subgroup of $SU(4)_R$. The suitability of this choice lies into the fact that it is the only one that leads to the desired $\mathcal{N} = 1$ supersymmetric models [56] (with $U(1)_R$ R-symmetry). Since the particles that consist the spectrum of the theory belong to different representations of $SU(4)_R$, it is expected that $\mathbb{Z}_3$ will act non-trivially on them. In the case of gauge and gaugino fields, the action of $\mathbb{Z}_3$ is trivial, since they are singlets under $SU(4)_R$. On the other hand, scalars and fermions will transform non trivially under the above action. Specifically, as far as the matter fields is concerned, since they are not invariant under a gauge transformation, $\mathbb{Z}_3$ acts on their gauge indices, too. Therefore, the orbifold filters this way the particle spectrum and the derived theory contains the particles which are invariant under the combined $\mathbb{Z}_3$ action on both the geometric[9] and gauge indices [61].

So - after the orbifold projection - the gauge group of the initial theory breaks down to the group $H = SU(N) \times SU(N) \times SU(N)$ with both scalar and fermionic fields transforming under the same representation $3 \cdot \left((N, \bar{N}, 1) + (\bar{N}, 1, N) + (1, N, \bar{N}) \right)$, a result that demonstrates the presence of the $\mathcal{N} = 1$ remnant supersymmetry. The chiral supermultiplet, which fermions and scalars share, is an anomaly free representation of H.

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[8]Ending up with mirror fermions does not forbid phenomenological contact [79], however exactly chiral fermions are preferred.

[9]In case of ordinary reduction of a 10-dimensional $\mathcal{N} = 1$ SYM theory, one obtains an $\mathcal{N} = 4$ SYM theory in four dimensions with a global $SU(4)_R$ symmetry which is identified with the tangent space $SO(6)$ of the extra dimensions [16, 17].

Besides the particle spectrum of the projected theory, it is necessary to determine its superpotential, which is derived from the superpotential of the initial $\mathcal{N} = 4$ SYM theory [66]

$$W_{\mathcal{N}=4} = \epsilon_{ijk}\mathrm{Tr}(\Phi^i \Phi^j \Phi^k)\,, \tag{26}$$

where the Φ's are chiral superfields. The above structure remains the same after the projection, but it encodes only the surviving fields of the $\mathcal{N} = 1$ theory that passed the orbifold filtering

$$W_{\mathcal{N}=1}^{(proj)} = \sum_I \epsilon_{ijk} \Phi_{I,I+a_i}^i \Phi_{I+a_i,I+a_i+a_j}^j \Phi_{I+a_i+a_j,I}^k\,. \tag{27}$$

The next step is to find the scalar potential of the projected thepry. This can be achieved by extracting it from the above superpotential, $W_{\mathcal{N}=1}^{(proj)}$. Therefore, the scalar potential is

$$V_{\mathcal{N}=1}^{proj}(\phi) = \frac{1}{4}\mathrm{Tr}\left([\phi^i,\phi^j]^\dagger[\phi^i,\phi^j]\right)\,, \tag{28}$$

where, ϕ^i are the scalar component fields of the chiral superfield Φ^i. Unfortunately, the minimization of $V_{\mathcal{N}=1}^{proj}(\phi)$ is achieved only by vanishing vevs of the fields, therefore, it is necessary to modify it in order that solutions which could be interpreted as vacua of a Non-commutative geometry to be emerged. So, addressed to this direction, $\mathcal{N} = 1$ soft supersymmetric terms of the form[10]

$$V_{SSB} = \frac{1}{2}\sum_i m_i^2 \phi^{i\dagger}\phi^i + \frac{1}{2}\sum_{i,j,k} h_{ijk}\phi^i\phi^j\phi^k + h.c. \tag{29}$$

are inserted into $V_{\mathcal{N}=1}^{proj}(\phi)$, where $h_{ijk} = 0$, unless $i + j + k = 0\mathrm{mod}3$. It is important to refer that the introduction of the SSB terms does not cause embarrassment, since an SSB sector is indispensable for a model aspired to be realistic, see e.g. [68].

So, the total scalar potential is

$$V = V_{\mathcal{N}=1}^{proj} + V_{SSB} + V_D\,, \tag{30}$$

where the term V_D represents the D-terms of the theory, which are given by

$$V_D = \frac{1}{2}D^2 = \frac{1}{2}D^I D_I\,, \tag{31}$$

---

[10]Only purely scalar SSB terms will be inserted into the scalar potential. Of course, other SSB terms have to be included in order to obtain the full SSB sector [68], however it is not necessary for our purposes.

where $D^I = \phi_i^\dagger T^I \phi^i$ and T^I are the generators of the chiral multiplets - in the representation they belong. Fixing the parameters of the (29) to $m_i^2 = 1$, $h_{ijk} = \epsilon_{ijk}$, the total scalar potential turns to be

$$V = \frac{1}{4}(F^{ij})^\dagger F^{ij} + V_D \,, \tag{32}$$

where F^{ij} is[11]

$$F^{ij} = [\phi^i, \phi^j] - i\epsilon^{ijk}(\phi^k)^\dagger \,. \tag{33}$$

Obviously, the first term of (32) is positive definite, which means that the global minimum of the potential is obtained if

$$[\phi^i, \phi^j] = i\epsilon_{ijk}(\phi^k)^\dagger \,, \quad \phi^i(\phi^j)^\dagger = R^2 \,, \tag{34}$$

with $[R^2, \phi^i] = 0$. It seems that the fuzzy sphere underlies in the above equations, so it just remains to designate it. This will arise by defining the untwisted fields $\tilde{\phi}^i$ as $\phi^i = \Omega\tilde{\phi}^i$, with $\Omega \neq 1$, satisfying the following relations

$$\Omega^3 = 1 \,, \quad [\Omega, \phi^i] = 0 \,, \quad \Omega^\dagger = \Omega^{-1} \,, \quad (\tilde{\phi}^i)^\dagger = \tilde{\phi}^i \;\Leftrightarrow\; (\phi^i)^\dagger = \Omega\phi^i \,. \tag{35}$$

Now, it is clear that (34) reduce to the ordinary fuzzy sphere relation generated by $\tilde{\phi}^i$

$$[\tilde{\phi}^i, \tilde{\phi}^j] = i\epsilon_{ijk}\tilde{\phi}^k \,, \quad \tilde{\phi}^i\tilde{\phi}^i = R^2 \,. \tag{36}$$

This demonstrates the reason why the Non-commutative space that generates the ϕ^i is called twisted fuzzy sphere, $\tilde{S}_N^2$. The fact that the above structure is valid only for $\mathbb{Z}_3$, poses it as the unique choice as the appropriate cyclic group for the orbifold projection.

A configuration of the twisted fields, ϕ^i, that satisfy (34) is $\phi^i = \Omega(\mathbf{1}_3 \otimes \lambda_{(N)}^i)$, where $\lambda_{(N)}^i$ are the generators in the N-dimensional irreducible representation of $SU(2)$ and Ω is the matrix described by the following relations:

$$\Omega = \Omega_3 \otimes \mathbf{1}_N \,, \quad \Omega_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \,, \quad \Omega^3 = \mathbf{1} \,. \tag{37}$$

According to the transformation $\phi^i = \Omega\tilde{\phi}^i$, the "off-diagonal" orbifold sectors are expressed in block-diagonal form

$$\phi^i = \begin{pmatrix} 0 & (\lambda_{(N)}^i)_{(N,\bar{N},1)} & 0 \\ 0 & 0 & (\lambda_{(N)}^i)_{(1,N,\bar{N})} \\ (\lambda_{(N)}^i)_{(\bar{N},1,N)} & 0 & 0 \end{pmatrix} = \Omega \begin{pmatrix} \lambda_{(N)}^i & 0 & 0 \\ 0 & \lambda_{(N)}^i & 0 \\ 0 & 0 & \lambda_{(N)}^i \end{pmatrix} \,. \tag{38}$$

---

[11]The V_D term settles for a change on the radius of the sphere, in accordance to the ordinary fuzzy sphere case [35, 69, 37]

It is clear that the (untwisted) fields, $\tilde{\phi}^i$, that generate the ordinary fuzzy sphere, have taken a block-diagonal form. Each block separately can be regarded as an ordinary fuzzy sphere, since the corresponding commutation relations (36) they separately satisfied. Therefore, the configuration (38) could be interpreted as three separate fuzzy spheres (branes), embedded with relative angles $2\pi/3$. Rephrasing, the solution ϕ^i is equivalent to three fuzzy spheres which conform with the orbifolding. In a few words, the global minimum of the scalar potential - at least for a fixed range of parameters - is achieved by a twisted fuzzy sphere. So, the F^{ij} that was defined in (33), could be considered as the field strength on the spontaneously generated fuzzy extra dimensions.

Let us now examine the potential's vacuum from a geometric point of view. Fixing the parameters, the potential gets minimized by a twisted fuzzy sphere solution

$$\phi^i = \Omega(\mathbf{1}_3 \otimes (\lambda^i_{(N-n)}) \oplus 0_n) \,, \tag{39}$$

where 0_n is the $n \times n$ matrix with zero entries. This non-vanishing vacuum - a vacuum considered as $\mathbb{R}^4 \times \tilde{S}^2_N$ with a twisted fuzzy sphere in the ϕ^i coordinates - breaks the gauge symmetry, $SU(N)^3$ down to $SU(n)^3$.

The fluctuations of the scalar fields around the vacuum can be understood by considering the transformation, $\phi^i = \Omega\tilde{\phi}^i$. In the non-twisted case, fluctuations around the vacuum describe scalar and gauge fields on S^2_N [54, 63], which gain mass from the $\mathbb{R}^4$ point of view. The (38) exhibits that the matrix Ω maps the twisted fuzzy sphere into three ordinary fuzzy spheres as three $N \times N$ blocks are diagonally embedded into the original $3N \times 3N$ matrix. Therefore, all fluctuations could be considered as fields on the three untwisted fuzzy spheres

$$\phi^i = \tilde{\Omega}(\lambda^i_{(N)} + A^i) = \begin{pmatrix} \lambda^i_{(N)} + A^i & 0 & 0 \\ 0 & \omega(\lambda^i_{(N)} + A^i) & 0 \\ 0 & 0 & \omega^2(\lambda^i_{(N)} + A^i) \end{pmatrix},$$

$$\tag{40}$$

as well as specific massive off-diagonal states which cyclically connect these spheres. The field strength F^{ij}, (33), converts to

$$F^{ij} = [\phi^i, \phi^j] - i\epsilon^{ijk}(\phi^k)^\dagger = \Omega^2\left([\tilde{\phi}^i, \tilde{\phi}^j] - i\epsilon^{ijk}\tilde{\phi}^k\right) \,, \tag{41}$$

that is the field strength on an untwisted fuzzy sphere. Thus, at intermediate energy scales, the vacuum can be interpreted as $\mathbb{R}^4 \times S^2_N$ with three untwisted fuzzy spheres in the $\tilde{\phi}^i$ coordinates. The three spheres do not mix, due to the lack of off-diagonal entries, due to the orbifold projection. As in [35, 36, 37], because of the Higgs effect, fermions and gauge fields decompose to a finite Kaluza-Klein tower of massive modes on S^2_N resp. $\tilde{S}^2_N$, as well as a massless sector.

6. The $SU(3)_c \times SU(3)_L \times SU(3)_R$ chiral model

Working in the above context, three (minimal and anomaly free) models emerge, however we will focus on the most interesting one. For all models the initial theory is the same, i.e. a $\mathcal{N} = 1$ SYM 4-dimensional $SU(3N)$ gauge theory whose field spectrum was listed in the previous section. Then it follows the realization of the orbifold projection of the theory, embedding - as we have already noted - the $\mathbb{Z}_3$ into the $SU(3)$ subgroup of $SU(4)_R$. After the projection, the initial gauge group breaks to the $\mathcal{N} = 1$ $SU(N)^3$ and the fields of the theory are accommodated into chiral representations of the gauge group. More specifically, there are three families transforming as

$$(N, \bar{N}, 1) + (\bar{N}, 1, N) + (1, N, \bar{N}) \tag{42}$$

under the gauge group $SU(N)^3$. Of course, due to the different ways the initial gauge group $SU(3N)$ is spontaneously broken, we end up with different unification groups $SU(4) \times SU(2) \times SU(2)$, $SU(4)^3$ and $SU(3)^{3\,12}$.

Let us now focus on the very interesting trinification group $SU(3)_c \times SU(3)_L \times SU(3)_R$ [70, 71] (see also [72, 73, 74, 75, 76]; for a string theory approach see [77]). At first, we need to decompose the integer N as $N = n + 3$ and then - for the $SU(N)$ - we consider the embedding

$$SU(N) \supset SU(n) \times SU(3) \times U(1) \,, \tag{43}$$

from which it is follows that the embedding for the total gauge group $SU(N)^3$ is

$$SU(N)^3 \supset SU(n) \times SU(3) \times SU(n) \times SU(3) \times SU(n) \times SU(3) \times U(1)^3 \,. \tag{44}$$

The three $U(1)$ factors[13] are ignored and according to the above decomposition, the representations (42) decompose (44), (after reordering the factors) as

$$SU(n) \times SU(n) \times SU(n) \times SU(3) \times SU(3) \times SU(3) \,,$$
$$(n, \bar{n}, 1; 1, 1, 1) + (1, n, \bar{n}; 1, 1, 1) + (\bar{n}, 1, n; 1, 1, 1) + (1, 1, 1; 3, \bar{3}, 1)$$
$$+ (1, 1, 1; 1, 3, \bar{3}) + (1, 1, 1; \bar{3}, 1, 3) + (n, 1, 1; 1, \bar{3}, 1) + (1, n, 1; 1, 1, \bar{3})$$
$$+ (1, 1, n; \bar{3}, 1, 1) + (\bar{n}, 1, 1; 1, 1, 3) + (1, \bar{n}, 1; 3, 1, 1) + (1, 1, \bar{n}; 1, 3, 1) \,. \tag{45}$$

So, judging from the decomposition (43), the gauge group is broken to $SU(3)^3$. The surviving fields transform under the gauge group $SU(3)^3$, as

$$SU(3) \times SU(3) \times SU(3) \,, \tag{46}$$
$$((3, \bar{3}, 1) + (\bar{3}, 1, 3) + (1, 3, \bar{3})) \,, \tag{47}$$

---

[12]Similar approaches have been examined in the YM matrix models framework [78], but they deprived of phenomenological viability.

[13]These factors decouple at the low energy sector of the theory due to a possible gaining of mass by the Green-Schwarz mechanism [67].

which correspond to the desired chiral representations of the trinification group. The transformation of quarks and leptons (only for the first family but it is similar for the other two) under the gauge group $SU(3)_c \times SU(3)_L \times SU(3)_R$ is

$$
q = \left(\begin{array}{ccc} d & u & h \\ d & u & h \\ d & u & h \end{array} \right) \sim (3, \bar{3}, 1) \,, \quad
q^c = \left(\begin{array}{ccc} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{array} \right) \sim (\bar{3}, 1, 3) \,,
$$

$$
\lambda = \left(\begin{array}{ccc} N & E^c & \mathbf{v} \\ E & N^c & e \\ \mathbf{v}^c & e^c & S \end{array} \right) \sim (1, 3, \bar{3}) \,, \tag{48}
$$

respectively.

It is crucial to mention that this theory could be upgraded to a two-loop finite theory (see reviews [80, 81, 82, 51]) and furthermore could make testable predictions [51].

Moreover, the fuzzy orbifold mechanism can be used to break spontaneously the unification gauge group down to MSSM and then, in turn, to the $SU(3)_c \times U(1)_{em}$. Concluding, it is useful to sum up the general idea of the theoretical model. At the very high-scale regime, there is an unbroken renormalizable gauge theory. After the spontaneous symmetry breaking, the resulting gauge theory we are led to, is an $SU(3)^3$ GUT, accompanied by a finite tower of massive Kaluza-Klein modes. Finally, in the low scale regime, the trinification group $SU(3)^3$ breaks down to the MSSM. Thus, we conclude with the statement that fuzzy extra dimensions can be used to construct chiral, renormalizable and phenomenologically viable field-theoretical models.

A natural extension of the above ideas and methods have been reported in ref [83] (see also [84]), realized in the context of Matrix Models (MM). At a fundamental level, the MMs introduced by Banks-Fischler-Shenker-Susskind (BFSS) and Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT), are supposed to provide a non-perturbative definition of M-theory and type IIB string theory respectively [30, 85]. On the other hand, MMs are also useful laboratories for the study of structures which could be relevant from a low-energy point of view. Indeed, they generate a plethora of interesting solutions, corresponding to strings, D-branes and their interactions [30, 86], as well as to non-commutative/fuzzy spaces, such as fuzzy tori and spheres [87]. Such backgrounds naturally give rise to non-abelian gauge theories. Therefore, it appears natural to pose the question whether it is possible to construct phenomenologically interesting particle physics models in this framework as well. In addition, an orbifold MM was proposed by Aoki-Iso-Suyama (AIS) in [88] as a particular projection of the IKKT model, and it is directly related to the construction described above in which fuzzy extra dimensions arise with trinification gauge theory [38]. By $\mathbb{Z}_3$ - orbifolding, the original symmetry of the IKKT matrix model with matrix size $3N \times 3N$ is generally reduced from $SO(9, 1) \times U(3N)$ to $SO(3, 1) \times U(N)^3$. This model is chiral and has $D = 4$, $\mathcal{N} = 1$ supersymmetry of Yang-Mills type as well as an inhomogeneous supersymmetry

specific to matrix models. The $\mathbb{Z}_3$ - invariant fermion fields transform as bi-fundamental representations under the unbroken gauge symmetry exactly as in the constructions described above. In the future we plan to extend further the studies initiated in refs [83, 84] in the context of orbifolded IKKT models.

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