Conformal invariance for scalar and Dirac particles in Riemannian spacetimes: New results

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Abstract

Conformal symmetry properties of pointlike scalar and Dirac particles (Higgs boson and all leptons) in Riemannian spacetimes in the presence of electromagnetic interactions are considered. A Hermitian form of the Klein-Gordon equation for a pointlike scalar particle in an arbitrary n-dimensional Riemannian spacetime is obtained. New conformal symmetries of initial and Hermitian forms of this equation are ascertained. In the above spacetime, general Hamiltonians in the generalized Feshbach-Villars and Foldy-Wouthuysen representations are derived. The conformal-like transformation conserving these Hamiltonians is found. Corresponding conformal symmetries of a Dirac particle are determined. It is proven that all conformal symmetries remain unchanged by an inclusion of electromagnetic interactions.

1. Introduction

A determination of symmetry properties of elementary particles is one of the most important problems of contemporary particle physics. Symmetries of basic relativistic wave equations describing pointlike particles with spin 0 (Higgs boson) and 1/2 (all leptons) retain an important place among these properties. Intensive studies of such symmetries have been started fifty years ago from the seminal work by Penrose [1]. He has discovered the conformal invariance of the covariant Klein-Gordon (KG) equation [2] for a massless scalar particle in a Riemannian spacetime and has supplemented this equation with a term describing a nonminimal coupling to the scalar curvature. Chernikov and Tagirov [3] have involved the case of a nonzero mass and n-dimensional Riemannian spacetime. The inclusion of the additional Penrose-Chernikov-Tagirov (PCT) term has been argued for both massive and massless particles [3]. Accioly and Blas [4] have performed the exact Foldy-Wouthuysen (FW) transformation for a massive spin-0 particle in static spacetimes and have found new telling arguments in favor of the predicted coupling to the scalar curvature. A derivation of the relativistic FW Hamiltonian is very important for a comparison of gravitational (and

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inertial) effects in classical and quantum gravity because the FW representation restores Schrödinger-like forms of Hamiltonians and equations of motion. These forms are convenient for finding a semiclassical approximation and a classical limit of relativistic quantum mechanics (see Refs. [5, 6, 7] and references therein).

However, the transformation method used in Ref. [4] is applicable neither to massless particles nor to nonstatic spacetimes. To find a specific manifestation of the conformal invariance in the FW representation which takes place just for massless particles, the generalized Feshbach-Villars (GFV) transformation [8] applicable for such particles has been performed [9]. The subsequent relativistic FW transformations has made it possible to derive the FW Hamiltonians for the both massive and massless scalar particles in general noninertial frames and stationary gravitational fields. The new manifestation of the conformal invariance for massless particles consisting in the conservation of the FW Hamiltonian and the FW wave function has been discovered. New exact FW Hamiltonians have been obtained for both massive and massless scalar particles in general static spacetimes and in frames rotating in the Kerr field approximated by a spatially isotropic metric. The high precision expression for the FW Hamiltonian has been derived in the general case. It has been also shown that conformal transformations change only such terms in the FW Hamiltonians which are proportional to the particle mass m.

In the present work, we consider the much more general problem of scalar and Dirac particles in arbitrary gravitational (noninertial) and electromagnetic fields and find (on a quantum-mechanical level) new symmetry properties relative to conformal transformations not only in the FW representation but also in initial representations. These properties are attributed to all known pointlike scalar and Dirac particles (Higgs boson and leptons) and also to the hypothetic pseudoscalar axion.

We denote world and spatial indices by Greek and Latin letters, respectively. The signature is $(+ - --)$, the Ricci scalar curvature is defined by $R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}{}_{\mu\alpha\nu}$, where $R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta\Gamma^{\alpha}{}_{\mu\nu} - \dots$ is the Riemann curvature tensor. We use the system of units $\hbar = 1,\ c = 1$.

2. Hermitian form of the Klein-Gordon equation and conformal symmetry for a pointlike scalar particle

The covariant KG equation with the additional PCT term [1, 3] describing a scalar particle in a n-dimensional Riemannian spacetime is given by

$$(\Box + m^2 - \lambda R)\psi = 0, \quad \Box \equiv \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu. \tag{1}$$

Minimal (zero) coupling corresponds to $\lambda = 0$, while the PCT coupling is defined by $\lambda = (n-2)/[4(n-1)]$ [3]. The sign of the Penrose-Chernikov-Tagirov term depends on the definition of R. For noninertial (accelerated and rotating) frames, the spacetime is flat and $R = 0$.

For a *massless* particle, the conformal transformation

$$\widetilde{g}_{\mu\nu} = O^{-2}g_{\mu\nu} \tag{2}$$

conserves the form of Eq. (1) but changes the wave function and the operators acting on it [1, 3]:

$$\Box - \lambda R = O^{-\frac{n+2}{2}}(\widetilde{\Box} - \lambda\widetilde{R})O^{\frac{n-2}{2}}, \quad \widetilde{\psi} = O^{\frac{n-2}{2}}\psi. \tag{3}$$

To specify symmetry properties of the initial KG equation (1), it is instructive to present it in the Hermitian form. Amazingly, this can be achieved with the simple nonunitary transformation

$$\psi = f^{-1}\Phi, \quad f = \sqrt{g^{00}\sqrt{-g}}, \quad g = \det g_{\mu\nu}. \tag{4}$$

Since $\widetilde{g} = O^{-2n}g$, Φ is invariant relative to the conformal transformation (2). After the transformation (4), we multiply the obtained equation by the factor f/g^{00} and come to the Hermitian form of the KG equation [10]:

$$\left(\frac{1}{f}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\frac{1}{f} + \frac{m^2}{g^{00}} - \frac{\lambda R}{g^{00}}\right)\Phi = 0. \tag{5}$$

The use of Eqs. (2)–(4) shows that Eq. (5) is conformally invariant for a massless particle. However, it is not conformally invariant for a massive one. To determine its conformal symmetry in the latter case, it is sufficient to find a physical quantity which substitution for m restores the conformal invariance of Eq. (5). For this purpose, we can use the quantity m' which is equal to m in the initial spacetime and takes the form

$$\widetilde{m'} = Om' \tag{6}$$

after the conformal transformation (2). The equation obtained from Eq. (5) with the substitution of m' for m,

$$\left(\frac{1}{f}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\frac{1}{f} + \frac{m'^2}{g^{00}} - \frac{\lambda R}{g^{00}}\right)\Phi = 0, \tag{7}$$

is conformally invariant. While this equation does not describe a real particle and is not equivalent to Eq. (5), finding the appropriate substitution (6) determines the conformal symmetry of the suitable equation (5). The determination of new symmetry property for massive particles is rather important because the only discovered pointlike scalar particle, Higgs boson, is massive.

Thus, we can conclude that Eq. (5) is not changed by the conformal-like transformation

$$\widetilde{g}_{\mu\nu} = O^{-2}g_{\mu\nu}, \quad m \to m', \quad \widetilde{m'} = Om'. \tag{8}$$

In the general case, we can substitute any quantity satisfying Eq. (6) for m into Eq. (5).

We can now state the conformal symmetry of the initial KG equation (1). The substitution of m' for m makes the *changed* equation to be conformally invariant with the following properties:

$$\Box + m'^2 - \lambda R = O^{-\frac{n+2}{2}}(\widetilde{\Box} + \widetilde{m'}^2 - \lambda\widetilde{R})O^{\frac{n-2}{2}}, \quad \widetilde{\psi} = O^{\frac{n-2}{2}}\psi. \tag{9}$$

These properties establish the conformal symmetry of the covariant KG equation (1) and the specific form of its invariance relative to the conformal-like transformation (8).

The method of the FW transformation used in Ref. [9] is applicable to nonstationary spacetimes. However, only the stationary case has been considered in this work. To make a more general investigation of symmetry properties in the FW representation, we need to present Eq. (5) in another (equivalent) form.

Let us introduce the following denotations:

$$\Gamma^i = \sqrt{-g}g^{0i}, \quad G^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}. \tag{10}$$

Lengthy but straightforward calculations bring Eq. (5) to the form [10]

$$\left[(\partial_0 + \Upsilon)^2 + \partial_i \frac{G^{ij}}{g^{00}}\partial_j + \frac{m^2}{g^{00}} + \Lambda\right]\Phi = 0, \tag{11}$$

where

$$\begin{gathered}
\Upsilon = \tfrac{1}{2f}\left\{\partial_i, \Gamma^i\right\}\tfrac{1}{f} = \tfrac{1}{2}\left\{\partial_i, \tfrac{g^{0i}}{g^{00}}\right\}, \\
\Lambda = -\tfrac{f_{,0\,,0}}{f} - \left(\tfrac{g^{0i}}{g^{00}}\right)_{,i}\tfrac{f_{,0}}{f} - 2\tfrac{g^{0i}}{g^{00}}\tfrac{f_{,0\,,i}}{f} - \left(\tfrac{g^{0i}}{g^{00}}\right)_{,0}\tfrac{f_{,i}}{f} \\
-\tfrac{1}{2}\left(\tfrac{g^{0i}}{g^{00}}\right)_{,0\,,i} - \tfrac{1}{2f^2}\left(\tfrac{g^{0i}}{g^{00}}\right)_{,i}\Gamma^j_{\ ,j} - \tfrac{g^{0i}}{2f^2g^{00}}\Gamma^j_{\ ,j\,,i} \\
+\tfrac{1}{4f^2}\left(\Gamma^i_{\ ,i}\right)^2 - \left(\tfrac{G^{ij}}{g^{00}}\right)_{,i}\tfrac{f_{,j}}{f} - \tfrac{G^{ij}}{g^{00}}\tfrac{f_{,i\,,j}}{f} - \tfrac{\lambda R}{g^{00}}.
\end{gathered} \tag{12}$$

This form of the KG equation is also Hermitian and the wave function is not changed as compared with Eq. (7). The replacement of m with m' makes Eq. (11) to be conformally invariant. Therefore, Eq. (11) is invariant relative to the conformal-like transformation (8).

3. Conformal symmetries of Hamiltonians

To fulfil the successive GFV and Foldy-Wouthyusen transformations, we use the method developed in Ref. [8] and applied to the covariant KG equation in Ref. [9]. The original Feshbach-Villars method does not work for massless particles while its generalization [8] makes it possible to extend the method on such particles.

We introduce two new functions, ϕ and χ, as follows [8, 9]:

$$\Phi = \phi + \chi, \quad i\left(\partial_0 + \Upsilon\right)\Phi = N(\phi - \chi), \tag{13}$$

where N is an arbitrary nonzero real parameter. For the Feshbach-Villars transformation, it is definite and equal to the particle mass m. These functions form the two-component wave function in the GFV representation, $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$. Equations (11) and (13) result in (cf. Ref. [9])

$$\begin{gathered} i\frac{\partial\Psi}{\partial t} = \mathcal{H}\Psi, \quad \mathcal{H} = \rho_3\frac{N^2+T}{2N} + i\rho_2\frac{-N^2+T}{2N} - i\Upsilon, \\ T = \partial_i\frac{G^{ij}}{g^{00}}\partial_j + \frac{m^2}{g^{00}} + \Lambda, \end{gathered} \tag{14}$$

where $\mathcal{H}$ is the GFV Hamiltonian and ρ_i $(i = 1, 2, 3)$ are the Pauli matrices. Equation (14) is exact.

For a massless particle, this Hamiltonian is not changed by the conformal transformation (2). In the general case, it is invariant relative to the conformal-like transformation (8).

The general methods developed in Refs. [7, 8, 6] allow us to perform the FW transformation of the Hamiltonian (14) for a relativistic particle in external fields. These methods are iterative. The initial Hamiltonian can be presented in the general form

$$\mathcal{H} = \rho_3\mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \rho_3\mathcal{M} = \mathcal{M}\rho_3, \quad \rho_3\mathcal{E} = \mathcal{E}\rho_3, \quad \rho_3\mathcal{O} = -\mathcal{O}\rho_3, \tag{15}$$

where $\mathcal{E}$ and $\mathcal{O}$ denote the sums of even (diagonal) and odd (off-diagonal) operators, respectively. In the considered case, $[\mathcal{M}, \mathcal{O}] = 0$,

$$\mathcal{M} = \frac{N^2+T}{2N}, \quad \mathcal{E} = -i\Upsilon, \quad \mathcal{O} = i\rho_2\frac{-N^2+T}{2N}, \tag{16}$$

and the transformation operator found in Ref. [7] reduces to the form [8, 9]

$$U = \frac{\epsilon + N + \rho_1(\epsilon - N)}{2\sqrt{\epsilon N}}, \quad \epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2} = \sqrt{T}. \tag{17}$$

This operator is ρ_3-pseudounitary ($U^\dagger = \rho_3 U^{-1}\rho_3$).

It is important that the Hamiltonian obtained as a result of this transformation does not depend on N [8]:

$$\begin{gathered} \mathcal{H}' = \rho_3\epsilon + \mathcal{E}' + \mathcal{O}', \quad \rho_3\mathcal{E}' = \mathcal{E}'\rho_3, \quad \rho_3\mathcal{O}' = -\mathcal{O}'\rho_3, \\ \mathcal{E}' = -i\Upsilon + \frac{1}{2\sqrt{\epsilon}}\left[\sqrt{\epsilon}, [\sqrt{\epsilon}, \mathcal{F}]\right]\frac{1}{\sqrt{\epsilon}}, \\ \mathcal{O}' = \rho_1\frac{1}{2\sqrt{\epsilon}}[\epsilon, \mathcal{F}]\frac{1}{\sqrt{\epsilon}}, \quad \mathcal{F} = -i\partial_0 - i\Upsilon. \end{gathered} \tag{18}$$

This shows a self-consistency of the used transformation method. The *exact* intermediate Hamiltonian (18) describes massive and massless particles and is not changed by the conformal-like transformation (8).

Next transformation [8] eliminates residual odd terms and leads to the final form of the *approximate* relativistic FW Hamiltonian:

$$\mathcal{H}_{FW} = \rho_3\epsilon - i\Upsilon - \frac{1}{2\sqrt{\epsilon}}\left[\sqrt{\epsilon}, [\sqrt{\epsilon}, (i\partial_0 + i\Upsilon)]\right]\frac{1}{\sqrt{\epsilon}}. \tag{19}$$

This final Hamiltonian is also invariant relative to the conformal-like transformation (8). As a rule, the relativistic FW Hamiltonian is expanded in powers of the Planck constant and it is useful when the de Broglie wavelength is much smaller than the characteristic distance [7]. In such a Hamiltonian, terms proportional to the zero and first powers of the Planck constant are determined exactly while less order terms are not specified (see Ref. [11]). As a result, the last term in Eq. (19) can be omitted if it is proportional to the second or higher orders of $\hbar$.

4. Inclusion of electromagnetic interactions

Fortunately, an inclusion of electromagnetic interactions does not leads to any significant complication of the above derivations. The initial covariant KG equation takes the form

$$\left[g^{\mu\nu}(\nabla_\mu + ieA_\mu)(\nabla_\nu + ieA_\nu) + m^2 - \lambda R\right]\psi = 0, \tag{20}$$

where ∇_μ is the covariant derivative and A_μ is the electromagnetic field potential. This equation is equivalent to the following one:

$$\left(\frac{1}{\sqrt{-g}}D_\mu\sqrt{-g}g^{\mu\nu}D_\nu + m^2 - \lambda R\right)\psi = 0, \tag{21}$$

where $D_\mu = \partial_\mu + ieA_\mu$. The nonunitary transformation (4) brings it to the Hermitian form corresponding to Eq. (5) [10]:

$$\left(\frac{1}{f}D_\mu\sqrt{-g}g^{\mu\nu}D_\nu\frac{1}{f} + \frac{m^2}{g^{00}} - \frac{\lambda R}{g^{00}}\right)\Phi = 0. \tag{22}$$

It is convenient to present this equation in the equivalent form (cf. Eq. (11)) [10]

$$\begin{gathered}\left[(D_0+\Upsilon')^2 + D_i\frac{G^{ij}}{g^{00}}D_j + \frac{m^2}{g^{00}} + \Lambda\right]\Phi = 0,\\ \Upsilon' = \frac{1}{2f}\left\{\partial_i,\Gamma^i\right\}\frac{1}{f} = \frac{1}{2}\left\{D_i,\frac{g^{0i}}{g^{00}}\right\},\\ T' = D_i\frac{G^{ij}}{g^{00}}D_j + \frac{m^2}{g^{00}} + \Lambda,\end{gathered} \tag{23}$$

where G^{ij} and Λ are defined by Eqs. (10) and (12), respectively.

A repeat of the transformation given above allows us to derive the Hamiltonian in the GFV representation:

$$\mathcal{H} = \rho_3\frac{N^2+T'}{2N} + \rho_2\frac{-N^2+T'}{2N} - i\Upsilon' + eA_0. \tag{24}$$

The FW transformation can be fulfilled with the operator (17) where $\epsilon = \sqrt{T'}$. The transformed operator $\mathcal{H}'$ is independent of N. The final approximate FW Hamiltonian is given by ($\epsilon = \sqrt{T'}$)

$$\mathcal{H}_{FW} = \rho_3\epsilon - i\Upsilon' + eA_0 - \frac{1}{2\sqrt{\epsilon}}\left[\sqrt{\epsilon},\left[\sqrt{\epsilon},(i\partial_0 + i\Upsilon' - eA_0)\right]\right]\frac{1}{\sqrt{\epsilon}}. \tag{25}$$

The last term in Eq. (25) can be omitted if it is proportional to the second or higher orders of $\hbar$ (see previous section).

All Hamiltonians obtained with the inclusion of electromagnetic interactions ($\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{H}_{FW}$) are invariant relative to the conformal-like transformation (8). The Hamiltonians are conformally invariant for a massless particle. Thus, this inclusion does not change the conformal symmetries of the Hamiltonians.

5. Conformal symmetry properties of Dirac particles

It is easier to determine the conformal symmetry properties of a pointlike Dirac particle. It has been established in Ref. [9] that the Dirac and FW Hamiltonians for a *massless* particle and the corresponding wave functions are invariant relative to the conformal transformation (2). The initial covariant Dirac equation is also conformally invariant for such a particle. The wave function of the massless particle in the conformally transformed metric (2) acquires the additional factor $O^{3/2}$ [9].

These results can be extended on massive particles. An analysis of the general relativistic equation for the Dirac Hamiltonian in arbitrary Riemannian spacetimes in the presence of an electromagnetic field (Eq. (2.21) in Ref. [12]) shows that this Hamiltonian is invariant relative to the conformal-like transformation (8). The FW transformation operator possesses the same property for both massive and massless particles. As a result, the FW Hamiltonian is also invariant relative to the conformal-like transformation in the general case. Finally, we find that the conformal-like transformation (8) of the covariant Dirac equation results in the following property of the wave function:

$$\widetilde{\Psi} = O^{3/2}\Psi. \tag{26}$$

Contrary to the conventional conformal invariance, this property is valid for any particle.

All properties stated in this section take place in the presence of electromagnetic interactions.

We can conclude that the previously ascertained similarity between massless scalar and Dirac particles in Riemannian spacetimes [9] exists for any pointlike particles and is not violated by electromagnetic interactions.

6. Summary

In the present work, new symmetry properties have been found for fundamental pointlike scalar and Dirac particles (Higgs boson and all leptons) in Riemannian spacetimes. All general results have been obtained in the presence of electromagnetic interactions. The KG equation for a pointlike scalar particle in an arbitrary n-dimensional Riemannian spacetime has been brought to the Hermitian form (5). This form is useful to derive the general Hamiltonians in the GFV and FW representations. The GFV Hamiltonians (14) and (24) are exact. The corresponding FW Hamiltonians (19) and (25) are approximate. They are expanded in powers of the

Planck constant and are useful when the de Broglie wavelength is much smaller than the characteristic distance. Nevertheless, these Hamiltonians are rather general. They cover the nonstationary case and can be applied for a relativistic particle in arbitrarily strong gravitational and inertial fields. In these Hamiltonians, terms proportional to the zero and first powers of the Planck constant are determined exactly while less order terms are not specified.

New conformal symmetries of the initial and Hermitian forms of the KG equation are ascertained. When the mass is replaced with any quantity m' satisfying the conformal transformation (6), the *changed* equations become conformally invariant. This defines the conformal symmetries of the conventional and Hermitian KG equations. The latter equation as well as the obtained Hamiltonians in the GFV and FW representations is invariant relative to the conformal-like transformation (8).

Corresponding conformal symmetries are also determined for both massive and massless Dirac particles. The Dirac and FW Hamiltonians are invariant relative to the conformal-like transformation (8). This transformation also defines the conformal symmetry of the initial Dirac equation for a massive particle. When m' defined by Eq. (6) is substituted for m, the Dirac wave function has the property (26).

It is proven that all conformal symmetries remain unchanged by an inclusion of electromagnetic interactions. Thus, the results obtained in the present study have allowed us to state the new general properties of conformal symmetry for pointlike scalar and Dirac particles (Higgs boson and all leptons) in Riemannian spacetimes in the presence of electromagnetic interactions.

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