# Another modification of the ACD sum rule

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Abstract

The Analytic Continuation by Duality (ACD) method is used to estimate dispersive integrals in low-energy QCD phenomenology and in technicolor models. The method uses a polynomial approximation of the $1/x$ function, usually a best-$L_p$ approximation. There are several sources of error in the ACD estimates which, along with the oscillatory behavior of the best-$Lp$ approximations, render them unreliable. The method is unstable, but it is still occasionally used in QCD phenomenology. Here we investigate a modification of the ACD method that has recently appeared by using a simple model spectrum. The modified ACD uses an approximation weighted towards the end of the interval, where the subasymptotic QCD is expected to be more reliable. For the case when only the first few terms of the Operator Product Expansion (OPE) are known, the modified method fails to reproduce the results expected for the simple model.

## 1. Introduction

The Analytic Continuation by Duality (ACD) is a method used to evaluate dispersive integrals in QCD using the first few terms of the Operator Product Expansion (OPE). ACD differs considerably from the usual QCD sum rules, but nevertheless it is often termed a “finite energy sum rule”. ACD is a fairly old method [2], used mainly in low-energy QCD phenomenology.

The ACD approximates the kernel of a dispersive integral by a polynomial on the real line, which allows the integration to be performed in the complex plane. As noted early [3], ACD is an ill-posed problem: small variations in the input parameters may lead to large variations in the results, which is discussed in one of our previous contributions [5]. Besides, we have found the ACD to be unstable and unreliable [4, 8]. The instability seems to be related to the oscillatory character of the best-$L_p$ polynomial approximations. It is difficult to investigate the reliability of the ACD method, thus, we have to be content with an analysis of its behavior on simple model spectra for which the exact value of the esimate is found from a dispersive integral and compared to the ACD-derived estimate.

Several modifications of the ACD method have been proposed. In one of the modifications [6] – called ‘simplified’ ACD here – the $1/s$ kernel is approximated by truncated Taylor series around the upper cut-off $R$. That

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simplified method is stable, but (contrary to claims) it yields only a lower bound for the estimates [7]. We have attempted a version of the ACD with an asymmetric weighting function used for the polynomial approximation, but it proved to be less reliable than the original ACD [9]. The ACD is claimed to require no phenomenological input, but in one of its versions the locations of the dominant resonances in the spectrum need to be known [10]. Therefore, if the resonances are not known that method depends on two new parameters, although the dependence is not necessarily so strong as to invalidate the entire approach [11].

A new version of of the ACD – called "modified ACD" in this contribution – introduces a weighting function straight into the dispersive integral [12] instead of using it only for the polynomial approximation. Mathematically, this is the most radical departure from the original ACD, which is the main motivation behind this contribution.

## 2. Original and Modified ACD Method

Finding the matrix element of a process with the vacuum polarization $\Pi(s)$ in the non-perturbative regime may lead to the dispersive integral

$$\int_{s_0}^{\infty} \frac{1}{s} \, \mathrm{Im} F(s) ds,$$

$F(s)$ being usually a simple function of $\Pi(s)$. The lower cut-off $s_0$ is the threshold of the process. However, $F(s)$ can be related to, e. g., a product of $\Pi(s)$ and another function, as we will see below.

The function $F(s)$ is analytic in the entire complex plane except on the branch cut along the real axis from $s_0$ so the Cauchy's integral formula gives

$$F(t) = \frac{1}{2\pi i} \oint_C \frac{F(s)}{s-t} ds, \quad (1)$$

where $C$ is the contour shown in Fig. 1.

The integral consists of two parts [5]:

$$F(t) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{F(s)}{s-t} ds + \frac{1}{\pi} \int_{s_0}^{R} \frac{\mathrm{Im} F(s)}{s-t-i\varepsilon} ds. \quad (2)$$

In the limit $t \to 0$ Eq. 2 becomes

$$F(0) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{F(s)}{s} ds + \frac{1}{\pi} \int_{s_0}^{R} \frac{\mathrm{Im} F(s)}{s} ds. \quad (3)$$

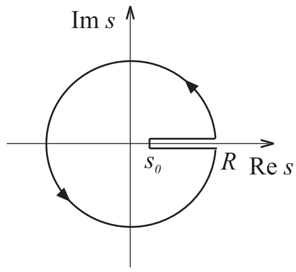

Figure 1: The contour of integration.

It can be shown that asymptotically $F(s) \sim 1/s$ so the integral over the circle $|s| = R$ vanishes as $R \to +\infty$. Therefore, in this limit we get the dispersion relation

$$\overline{F} \equiv F(0) = \frac{1}{\pi} \int_{s_0}^{+\infty} \frac{\mathrm{Im} F(s)}{s} ds. \quad (4)$$

In the original ACD, the function $F$ is simply related to the vacuum polarization $\Pi(s)$. In the modified ACD we take $F(s) = G(s)\rho(s)$, where $\rho(s)$ is a real continuous function obeying $\rho(0) = 1$ while $G(s)$ is simply related to $\Pi(s)$. Now instead of (3) we find

$$G(0) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{G(s)\rho(s)}{s} ds + \frac{1}{\pi} \int_{s_0}^{R} \frac{\mathrm{Im} G(s)\rho(s)}{s} ds \qquad (5)$$

and instead of (4)

$$\overline{G} \equiv G(0) = \frac{1}{\pi} \int_{s_0}^{+\infty} \frac{\mathrm{Im} G(s)\rho(s)}{s} ds. \qquad (6)$$

In contrast to the approach that leads to the dispersion relation (4), the ACD method makes the *second* integral in (3), i. e. in (5) for modified ACD, vanish. This is achieved by approximating the kernel $1/s$ by a polynomial. In the original ACD, the kernel is approximated by

$$p_N(s) = \sum_{n=0}^{N} a_n(N) s^n$$

and we find [5]

$$F(0) = \frac{1}{2\pi i} \oint_{|s|=R} \left[\frac{1}{s} - p_N(s)\right] F(s)\, ds + \frac{1}{\pi} \int_{s_0}^{R} \left[\frac{1}{s} - p_N(s)\right] \mathrm{Im} F(s)\, ds. \qquad (7)$$

Eq. (7) is exact. We try to make the second integral as small as possible; it is called "fit error". The fit error depends on $N$, on the interval $[s_0, R]$ and on the fit routine. Clearly, the minimization of the second integral in (5) is more involved as the function $\rho(s)$ is undetermined.

## 3. Approximation Routine for Modified ACD

The problem of minimization of the fit error is to find the norm that is to be minimized and to find an appropriate function $\rho(s)$. The approach taken in [12] is to divide the weight function into two parts: $\rho(s) = D_1(s)D_2(s)$. The two parts are actually found employing the ideas that have already been used within the original ACD. Since the (usually asymptotic) large-$s$ expansion tends to be better near the upper bound $R$ it makes sense to require that the function $D_2(s)$ and its derivatives vanish at $s = R$:

$$\left.\frac{d^k}{ds^k} D_2(s)\right|_{s=R} = 0, \qquad k = 0, \ldots, N_2. \qquad (8)$$

This was the idea of the simplified ACD [6], but in that case there was no (nontrivial) $D_1(s)$ so that version gave only a lower bound of the ACD-estimates. In fact [7],

$$D_2(s) = \left(1 - \frac{s}{R}\right)^{N_2+1}. \qquad (9)$$

The function $D_1(s)$ is taken to be the polynomial that minimizes the weighted $L_2$-norm

$$||d|| = \left[\int_{s_0}^{R} [d(s)]^2 D_2(s) ds\right]^{1/2}, \tag{10}$$

where

$$d(s) = \frac{D_1(s)}{s} = \frac{1}{s} - p_{N_1}(s) = \frac{1}{s} - \sum_{n=0}^{N_1} a_n(N_1, N_2) s^n, \tag{11}$$

With a change of variables $x = -ps + q$, where

$$p = \frac{2}{R - s_0}, \qquad q = \frac{R + s_0}{R - s_0}$$

the approximation problem is reduced to the minimization of the integral

$$||d'||^2 = \int_{-1}^{1} \left[\frac{1}{x-q} - \sum_{n=0}^{N_1} b_n(N_1, N_2) x^n\right]^2 (1+x)^{N_2+1} dx, \tag{12}$$

where $||d'||$ differs from the norm $||d||$ in (10) only by an overall factor. The change of variables has rescaled and *inverted* the interval: $s_0$ corresponds to $+1$ and $R$ to $-1$. There is also an overall factor $-p$:

$$\frac{1}{s} = \frac{-p}{x-q}.$$

It is straightforward to express $a_n$ through $b_n$ [9]:

$$a_n = \sum_{k=n}^{N} \binom{k}{i} (-p)^{n+1} q^{k-n} b_k. \tag{13}$$

Thus the problem is reduced to finding the coefficients $b_k$.

It is known [13] that the polynomials which minimize the integral (12) are special case of the Jacobi polynomials:

$$y_n(x) = P_n^{(0, N_2+1)}(x), \qquad n = 0, \ldots, N_1$$

and that they are orthogonal with the above weight:

$$\int_{-1}^{1} y_m(x) y_n(x) (1+x)^{N_2+1} dx = d_n^2 \delta_{mn},$$

where the constant $d_n = d_n(N_2)$ is obtained by specializing the general normalization factor for Jacobi polynomials [14]

$$d_n^2 = \frac{2^{N_2+2}}{2n + N_2 + 2}.$$

Hence we fit the function $1/(x-q)$ to truncated Fourier-Jacobi series:

$$\frac{1}{x-q} \approx \sum_{n=0}^{N_1} c_n y_n(x),$$

where $c_n$ are the Fourier-Jacobi coefficients. We have studied the general case of approximation with Jacobi polynomials in our previous contribution [9] so we need only to specialize the results to the present case. The Fourier-Jacobi coefficients are

$$c_n = \frac{1}{d_n^2} \int_{-1}^{1} \frac{y_n(x)(1+x)^{N_2+1}}{x-q} dx, \qquad q > 1. \tag{14}$$

The integral in (14) for Jacobi polynomials $P_n^{(0,N_2+1)}(x)$ can be expressed in terms of the Jacobi functions of the second kind [13]:

$$Q_n^{(0,N_2+1)}(q) = \frac{1}{2(q+1)^{N_2+1}} \int_{-1}^{1} \frac{P_n^{(0,N_2+1)}(x)(1+x)^{N_2+1}}{q-x} dx. \tag{15}$$

Eventually the coefficients are found to be

$$c_n(q) = -2\frac{Q_n^{(0,N_2+1)}(q)}{d_n^2}(q+1)^{N_2+1}, \qquad n = 0, \ldots, N_1. \tag{16}$$

Jacobi functions of the second kind are easily computable since they are corecursive with the Jacobi polynomials. It is necessary to find $Q_0$ and $Q_1$ to initialize the recursive procedure. Actually, the only non-polynomial function that appears in $Q_n$ is the one that is present in $Q_0$ and it is logarithmic [15]. In our case, the function is

$$\ln \frac{1+q}{1-q} = \ln \frac{R}{s_0}.$$

However, In most realistic applications $N_1 \leq 2$ so that it is not difficult to obtain the fits to $1/s$ directly from (10) and (11).

Finally, we find the function $\rho(s)$:

$$\rho(s) = D_1(s)D_2(s) = \left[1 - \sum_{n=0}^{N_1} a_n(N_1, N_2)s^{n+1}\right]\left(1 - \frac{s}{R}\right)^{N_2+1}. \tag{17}$$

The function $\rho(s)$ is a polynomial of order $N_1 + N_2 + 2$:

$$\rho(s) = \sum_{n=0}^{N_1+N_2+2} A_n(N_1, N_2)s^n, \tag{18}$$

which has been fully determined and enables us to derive the ACD estimate. Of course, not all of the $A_n$'s are independent (e. g. $A_0 = 1$ for all $N_1, N_2$); in fact, only $N_1 + 1$ $A_n$'s can be considered independent.

## 4. Derivation of the ACD Estimate

Let us assume that the large-$s$ expansion

$$G(s) = \sum_{m=1}^{M} \frac{h_m(s)}{s^m} + O\left(\frac{1}{s^{M+1}}\right) \tag{19}$$

is valid around the circle $|s| = R$. The expansion (19) may be OPE, which is an asymptotic series, the large-momentum expansion of a perturbative series or an exactly known vacuum polarization, which may be convergent series (the model considered in Sec. 5 belonging to the last case). For the subasymptotic QCD only a few terms of the OPE are known, thus in realistic applications $M \le 4$.

Returning to the main ACD formula (5) the ACD estimate reeds

$$G(0) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{G(s)\rho(s)}{s} ds + \Delta_{\text{fit}}, \tag{20}$$

where $\Delta_{\text{fit}}$ is the fit error.

The truncation of the series (19) introduces a further error and the neglect of the momentum dependence of the large-momentum expansion coefficients $h_m$ (which is not always necessary, however) an additional error [5]. We do not consider these two errors here since the fit error is the only error for the model of Sec. 5. The only non-zero contributions to the first integral in (5) come from the simple poles so that

$$G_{\text{ACD}} = \sum_{n=0}^{\min(N_1+N_2+2, M-1)} \hat{h}_{n+1} A_n(N_1, N_2). \tag{21}$$

If the spectral function is known exactly, the dispersion relation gives a reliable estimate of $\overline{G}$ so the comparison to $G_{\text{ACD}}$ will determine if the ACD estimate is reliable.

Apparently, the modified ACD introduces two integer parameters $N_1$ and $N_2$ as opposed to the original ACD which has only one (the order of the fit polynomial $N$). However, in most early applications $M \le 4$ so that looking at (21) we can restrict $N_1 + N_2$ to just 0 or 1. Therefore, there are essentially only three possibilities: 1) $N_1 = N_2 = 0$, 2) $N_1 = 0, N_2 = 1$ and 3) $N_1 = 1, N_2 = 0$. To be fair, one has to note that the actual application in [12] used an expansion with $M = 10$, which allowed to choose among many more $N_1$, $N_2$.

## 5. ACD Estimates for Model Spectra

In order to investigate the reliability of the ACD-derived results we use a simple model function $G(s)$ for which the dispersive integral (6) can be evaluated exactly. Our model contains two infinitely sharp ($\delta$-function) resonances – a single resonance of mass $m_V$ in the vector channel and a single resonance of mass $m_A$ in the axial channel. The parameters of the resonances are constrained by the condition that the spectra saturate the first and second Weinberg sum rules. We find

$$\mathrm{Im}G(s) = -\pi[f_V^2\delta(s-m_V^2) - f_A^2\delta(s-m_A^2)],$$

$$G(s) = \frac{f_V^2}{s-m_V^2+i\epsilon} - \frac{f_A^2}{s-m_A^2+i\epsilon}.$$

Clearly,

$$\overline{G} = G(0) = \frac{f_A^2}{m_A^2} - \frac{f_V^2}{m_V^2} \tag{22}$$

provided $s_0 < m_V, m_A < R$. The Weinberg sum rules give [5]

$$\overline{G} = (1+r)\frac{f^2}{m_V^2}, \tag{23}$$

where $r = m_V^2/m_A^2$ (for QCD $r \approx 0.4$). The large-momentum expansion is

$$G(s) = \sum_{n=0}^{+\infty} \frac{f_V^2 m_V^{2n} - f_A^2 m_A^{2n}}{s^{n+1}} = \sum_{n=0}^{+\infty} \frac{f^2 m_V^{2n}}{s^{n+1}} \frac{1-r^{1-n}}{1-r}.$$

Interestingly, in this model the term with $1/s^2$ is absent.

It is possible to use the more realistic model with finite-width resonances. However, our previous analysis indicated that the model yields quallitatively the same results hence we have not used it in our recent contributions.

In general, only a few terms of the OPE are known [1, 2] so in most applications $M \leq 4$, although there are exceptions. In order to check the sensitivity of the estimates to the location of the resonances we have calculated $G_{\mathrm{ACD}}$ not only for its QCD value $r = 0.4$, but also for $r = 0.3$ and $r = 0.5$. This is equivalent to varying the mass $m_A$ of the axial channel resonance with the mass $m_V$ of the vector channel resonance fixed. The results are shown in Table 1.

We have used only the lowest-order approximations with $N_1 + N_2 = 0, 1$. It is clear not only that the results are unsatisfactory, but also that increasing the order of the approximation made the results worse! The computationally much more demanding cases with $N_1+N_2 = 8$ (or so) could perhaps yield better results and will be the subject of further investigation.

Another negative surprise was the strong dependence on the lower cut-off $s_0$ (not shown on the table), much more severe than in our previous contribution [9]. That dependence is a known problem with the ACD [16, 8] and it is particularly serious for problems with no natural lower cut-off (such as the two-pion threshold) as was the case for the technicolor.

Table 1: $G_{\text{ACD}}$ in units of its exact value (23) for 3 values of the dimensionless parameter $r$. The cutoffs are $s_0 = 0.2m_V^2$ and $R = 5m_V^2$.

| $r$ | $N_1 = N_2 = 0$ | $N_1 = 0, N_2 = 1$ | $N_1 = 1, N_2 = 0$ |
|---|---|---|---|
| 0.3 | 0.266 | −0.042 | −0.489 |
| 0.4 | 0.364 | 0.077 | −0.340 |
| 0.5 | 0.405 | 0.158 | −0.200 |

## 6. Conclusions

The ACD is an old method of non-perturbative QCD with several known shortcomings. The modified ACD investigated in this contribution represents a major change from a mathematical point of view. It changes not only the fit routine of the polynomial approximation – which is crucial for the ACD, but also the way the ACD estimates are derived. The modified ACD may also be regarded as a generalization of the ACD method because it introduces two undetermined functions thus opening up new possibilities in the development of the method. In this contribution we have investigated only the concrete application of this concept proposed by the authors of the modified ACD.

The modified ACD weights the polynomial approximation towards the upper cut-off, which is expected to improve the estimates from a physical point of view, namely, the subasymptotic QCD improves with increasing momenta. Since the results are poor, it appears that the weighting is too severe, although we have checked the reliability only for the lowest-order approximations.

The dependence of the ACD-derived estimates on the lower cut-off is also a major problem in the modified ACD, even more severe than in the original ACD.

Hence the modified ACD does not appear to improve the method, but further investigation of this interesting approach is necessary. Firstly, one has to check the reliability and stability of the method for considerably higher orders of the approximation. Secondly, the modification of the method the ACD estimates are derived increases the freedom of choice of the approximation methods and should be further explored.

## References

[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nucl. Phys.* **B147** (1979) 385

[2] N. F. Nasrallah, N. A. Papadopoulos and K. Schilcher, *Phys. Lett.* **B113** (1982) 61; **B126** (1983) 379; *Z. Phys.* **C16** (1983) 323.

[3] I. Caprini and C. Verzegnassi, *Il Nuovo Cim.* **80A** (1984) 187.

[4] Siniša R. Ignjatović, *Analytic continuation by duality method in elementary particle physics*, PhD thesis, University of Cincinnati, 1997, and references therein.

[5] S. R. Ignjatović, in *Proc. IV Summer School in Modern Mathematical Physics*, SFIN (Belgrade) **20A**, 239 (2007).

[6] N. F. Nasrallah, *Phys. Rev.* **D63** (2001) paper 054028.

[7] S. R. Ignjatović, *Fortschr. Phys.* **56** (2008), 452.

[8] S. R. Ignjatović, T. Takeuchi, and L. C. R. Wijewardhana, *Phys. Rev.* **D61** (2000) paper 056006.

[9] S. R. Ignjatović, in *Proc. V Summer School in Modern Mathematical Physics*, SFIN (Belgrade) **22A**, 185 (2009).

[10] C. A. Dominguez, N. F. Nasrallah and K. Schilcher, JHEP (2008) 02 072; C. A. Dominguez, *Mod. Phys. Lett.* **A26**, (2011) 691.

[11] S. R. Ignjatović, *Rom J.. Phys.* **57** (2012), 857.

[12] M. Li and J. Liu, *Int. J. Mod. Phys. E* **21** (2012), paper 1250048.

[13] H. Bateman and A. Erdélyi, *Higher Transcedental Functions* v. 2, Mc-Graw Hill, New York, 1953.

[14] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, Boston, 1988.

[15] J. Wimp, P. McCabe and J. N. L. Connor, *J. Comp. Appl. Math.* **82** (1997) 447.

[16] M. Kramer et al., *Phys. Rev.* **D34** (1986) 2127.